# lecture

03 March 2021

RSS = Riemannian globally symm. space

**Proposition II.7** $M$ RSS, then $M$ is complete and $\mathrm{Iso}(M)^\circ$ acts transitively.

Last week completeness
Now transitivity.

**Lemma II.8** $M$ RSS, then the map $p \longmapsto s_p$ is continuous.

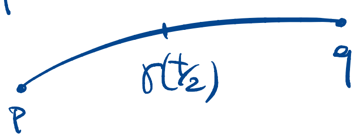
How do we use this:

$$M \times M \longrightarrow \mathrm{Iso}(M)$$
$$(p,q) \longmapsto s_p \circ s_q$$

Since $s_p^2 = \mathrm{Id}$ $\Rightarrow$ the image of this map is in $\mathrm{Iso}(M)^\circ$ ($M$ is connected) $\Rightarrow$ $s_p \circ s_q \in \mathrm{Iso}(M)^\circ$.

If $\gamma$ is a geod. from $p$ to $q$ with $\gamma(0)=p$, $\gamma(t)=q$ $\Rightarrow$

$$s_{\gamma(t/2)} \circ s_p(p) = q$$




Need to show the lemma.

**Exercise** $M$ RSS, $p \in M$, $K := \mathrm{Stab}_{\mathrm{Iso}(M)}(p)$. Then the orbit map

$$\mathrm{Iso}(M)/K \longrightarrow M$$
$$gK \longmapsto gp$$

is a homeomorphism. ☒

**Pf of lemma II.8**

Verify that $s_{gp} = g s_p g^{-1}$.

In fact, again by lemma II.2

$$g s_p g^{-1}(gp) = g s_p(p) = gp$$

$$d_{gp}(g s_p g^{-1}) = (d_p g)\underbrace{(d_p s_p)}_{-1}(d_{gp} g^{-1}) =$$

$$= -(d_p g)(d_{gp} g^{-1}) = -d_{gp}\mathrm{Id} = -1 = d_{gp} s_{gp}.$$

$K := \mathrm{Stab}_{\mathrm{Iso}(M)}(p)$

$$\begin{tikzcd}
gp \arrow[rrr] & & & s_{gp} = g s_p g^{-1} \\
M \arrow[rr] & & \mathrm{Iso}(M) & \\
\mathrm{Iso}(M)/K \arrow[u, shift left, "homeo."] \arrow[ur, "cont."] & & & \\
\mathrm{Iso}(M) \arrow[u] \arrow[uur] & & &
\end{tikzcd}$$

$$g \longmapsto gK \longmapsto gp, \qquad g \longmapsto s_{gp}$$

(1) $g \longmapsto g s_p g^{-1}$ continuous

(2) $g \longmapsto s_{gp}$ cont.

(3) $g \longmapsto s_{gp}$ descends to a map $gK \longmapsto s_{gp}$

$\Rightarrow$ $p \to s_p$ is the comp. of the inverse of the orbit map and the map in (3) $\Rightarrow$ cont. ☒

**Corollary II.9** $(M,g)$ RSS, $p \in M$, $K := \mathrm{Stab}_{\mathrm{Iso}(M)}(p)$. Then $K$ meets every conn. cpt. of $\mathrm{Iso}(M)$. In particular $\mathrm{Iso}(M)^\circ$ is open and of finite index in $\mathrm{Iso}(M)$.

**Pf** $g \in \mathrm{Iso}(M)$. Since $\mathrm{Iso}(M)^\circ$ acts trans. $\Rightarrow$ $\exists\, g_0 \in \mathrm{Iso}(M)^\circ$ s.t. $gp = g_0 p$ $\Rightarrow$ $\exists\, k \in K$ s.t. $g = g_0 k$ $\Rightarrow$ $K$ meets every conn. cpt. ($k \in g\,\mathrm{Iso}(M)^\circ$).

To see the second assertion, $\mathrm{Id} \in K^\circ \Rightarrow K^\circ \subset \mathrm{Iso}(M)^\circ$ thus the homo $\alpha : K \longrightarrow \mathrm{Iso}(M)/\mathrm{Iso}(M)^\circ$ factors through $K^\circ$ $\Rightarrow$

$$\Rightarrow K/K^\circ \longrightarrow \mathrm{Iso}(M)/\mathrm{Iso}(M)^\circ$$

By the first assertion this is onto

$$\Rightarrow |K/K^\circ| < \infty \Rightarrow |\mathrm{Iso}(M)/\mathrm{Iso}(M)^\circ| < \infty$$ ☒

**Thm II.10** $M$ RSS. Then $G := \mathrm{Iso}(M)$ has a Lie gp. structure compatible with the cpt. open top. and it acts smoothly on $M$. Moreover

$$\mathrm{Iso}(M)/K \longrightarrow M \text{ is a diffeo}$$

and $K$ contains no non-trivial normal subgps of $G$.

(Helgason IV.3.2 for a complete proof)

**Idea of the proof**

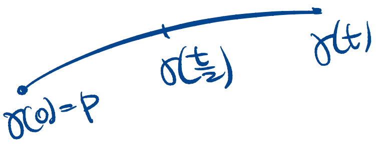
- We saw already that $K$ has a smooth structure given by the injection $K \hookrightarrow O(T_pM)$ which realizes $K$ as a closed sbgp of $O(T_pM)$ hence a Lie group.
- Let $\pi: G \longrightarrow M = G/K$. We are going to define a cont. local section $\phi: U \longrightarrow G$, where $U$ is a normal nbd of $p$, $\pi \circ \phi = id|_U$. $\Rightarrow \phi$ is a homeo onto its image (inj. and with cont. inverse $= \pi$). Thus we can define

$$\varphi: U \times K \longrightarrow \pi^{-1}(U)$$
$$(x,k) \longmapsto \phi(x)k$$

that is continuous, bijective with inverse given by $\varphi^{-1}: \pi^{-1}(U) \to U \times K$, $g \longmapsto (gp, \phi(gp)p) \Rightarrow \varphi^{-1}$ is a homeo between $\pi^{-1}(U) \ni Id$ and $U \times K$. The smooth structure on $G$ is given by the smooth structure on translates of $\pi^{-1}(U)$.

So, how do we define $\phi$.

Let $\gamma$ be a geodesic contained in $U$ s.t. $\gamma(0) = p \Rightarrow s_{\gamma(t/2)} \circ s_p(p) = \gamma(t)$

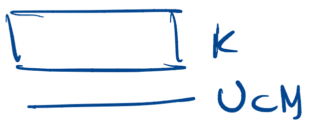



$\Rightarrow \phi: U \to G$ can be defined as $\phi(\gamma(t)) := s_{\gamma(t/2)} \circ s_p$.

By lemma II.8 this is continuous.

Finally $\nexists$ non-trivial subgps in $K$ that are normal in $G$. In fact if there were such a sbgp, it would act trivially on $M$. ☒

**Remark** $\pi: G \longrightarrow M \Rightarrow G$ is a principal bundle over $M$ with fiber $K \Rightarrow$ locally $G$ is a product

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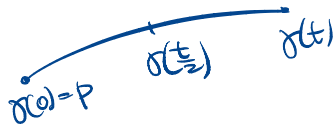


## Some concept of Riem. geom.
(Boothby, Do Carmo)

$M$ smooth mfd, $\pi: TM \to M$

A smooth v.f. is a section of $\pi$

$X : M \to TM$ s.t. $\pi \circ X = id$.

$Vect(M)$ = v.f. on $M$, is a $C^\infty(M)$-mod. with ptwise multipl.: $f \in C^\infty(M)$, $X$ v.f.

$$(f \cdot X)_p = f(p) X_p.$$

If $f \in C^\infty(M)$, $d_p f : T_pM \to T_{f(p)}M$ is the diff. Any $X \in Vect(M)$ acts on $C^\infty(M)$

$$(Xf)(p) = (d_pf)(X_p).$$

**Defn.** A connection on $M$ is a map $\nabla : Vect(M) \times Vect(M) \to Vect(M)$ s.t.

(1) $\nabla$ is $C^\infty(M)$-linear in $X$

$$\nabla_{fX + f'X'}(Y) = f\nabla_X Y + f' \nabla_{X'} Y$$

$\forall X, X', Y \in Vect(M)$, $f \in C^\infty(M)$

(2) $\nabla$ is $\mathbb{R}$-linear in $Y$

$$\nabla_X (aY + bY') = a \nabla_X Y + b\nabla_X Y'$$

$\forall X, Y, Y' \in Vect(M)$, $a, b \in \mathbb{R}$

(3) (Leibniz rule)

$$\nabla_X(fY + f'Y') = f\nabla_X Y + f'\nabla_X Y' + \boxed{(Xf)Y + (Xf')Y'}$$

$X, Y, Y' \in Vect(M)$, $f, f' \in C^\infty(M)$

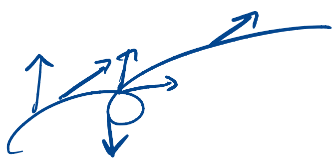
**Rk** $\nabla_X Y$ depends only on $X_p$ but on $Y$ in a nbd of $p$.

**Terminology** Let $I \subset \mathbb{R}$ be an interval. A map $f : I \to N$ into a smooth mfd is smooth if it the restr. of a smooth map defined on an open interval containing $I$.

**Defn.** Let $\gamma : I \to M$ be a smooth curve. A vector field along $\gamma$ is a smooth map

$X : I \to TM$ s.t.

$X(t) \in T_{\gamma(t)}M$ (not nec. tg. to $\gamma$!)



$Vect(\gamma^* TM)$ = vector space of vector fields along $\gamma$.

Let $\gamma : \mathbb{R} \to M$ be a smooth curve and $X \in Vect(\gamma^*TM)$.

$\nabla_{\dot\gamma(t)} X$ is the covariant derivative of $X$ along $\gamma$

**Defn.** Let $X \in Vect(\gamma^*TM)$ be a v.f. along a smooth curve $\gamma$. We say that $X$ is parallel if $\nabla_{\dot\gamma} X = 0$.

**Remark** $\gamma \subset \mathbb{R}^n$ and $X \in Vect(\gamma^*TM)$ we can decompose

$$T\mathbb{R}^n = \mathbb{R}\dot\gamma \oplus (\mathbb{R}\dot\gamma)^\perp.$$

Then $\nabla_{\dot\gamma} X = pr_{(\mathbb{R}\dot\gamma)^\perp}\left(\frac{dX}{dt}\right)$

$S^1 \subset \mathbb{R}^2$


$\nabla_{\dot\gamma} X \equiv 0$ even though $\left|\frac{dx}{dt}\right| \equiv 1$

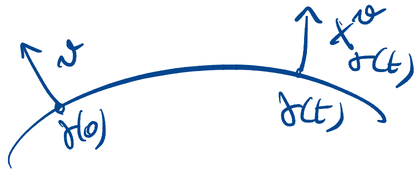
The same applies to the tg. vector along a great circle in $S^{n-1} \subset \mathbb{R}^n$, parametrized by arclength. In fact geodesics can be defined as curves $\gamma$ s.t. $\nabla_{\dot\gamma}\dot\gamma \equiv 0$

Rare: $Y \in \mathrm{Vect}(M)$ s.t.

$$(\nabla_X Y)_p = 0 \quad \forall p \in M$$

$\forall\, X \in \mathrm{Vect}(M)$.

Proposition II.11 $M$ diff. mfd, $\gamma \subset M$ smooth curve. Given $v \in T_{\gamma(0)}M$ $\exists!$ vector field $X^v \in \mathrm{Vect}(\gamma^* TM)$ parallel along $\gamma$, & such that $X^v_{\gamma(0)} = v$.




We can define the parallel transport along a curve $\gamma$ from $\gamma(0)$ to $\gamma(t)$.

$$P_{\gamma,[0,t]} : T_{\gamma(0)}M \longrightarrow T_{\gamma(t)}M$$

$$P_{\gamma,[0,t]}(v) := X^v_{\gamma(t)}.$$

Because of uniqueness

$$P_{\gamma,[t_1,t_2]} \circ P_{\gamma,[t_0,t_2]} = P_{\gamma,[t_0,t_2]}$$