Exercise Class 1: Symmetric Spaces

4.3.21

Plan for today:

I) Recap

II) Excursion: Homogeneous Spaces

I) Recap:

Geodesic symmetries.

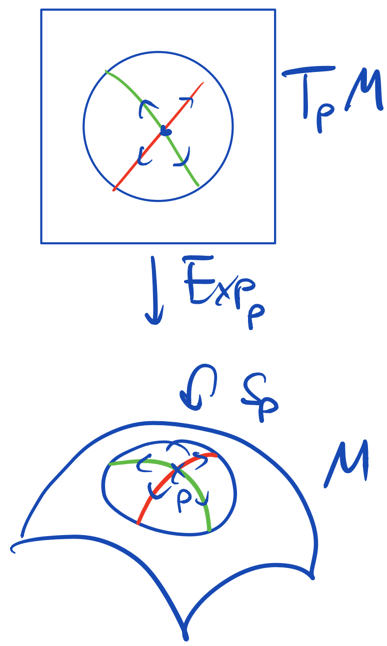
Defn. M a Riem. mfd. $p \in M$.

(a) M is Riemannian locally symmetric if for every $p \in M$ there exist a normal nbd $U \ni p$ and an isometry $s_p : U \to U$ s.t.

(1) $s_p^2 = \mathrm{Id}$

(2) p is the only fixed pt of s_p in U.

s_p is a geodesic symmetry

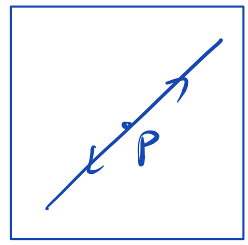


(b) M is Riem. globally symm. if $\forall p \in M$ s_p can be extended to M.

Examples:

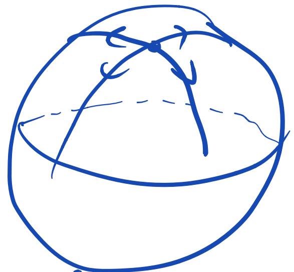
globally symmetric

(1) $\mathbb{R}^4$



s_p = "point reflection at p"

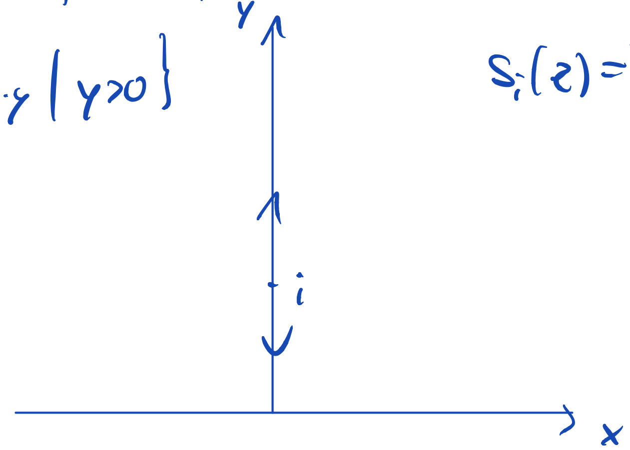
(2) S^4



hyperbolic plane/space

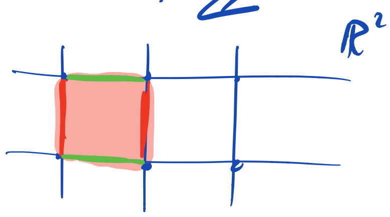
(3) $\mathbb{H}^2 = \{x + i\cdot y \mid y > 0\} \subseteq \mathbb{C}$

$s_i(z) = -\frac{1}{z}$ (?)

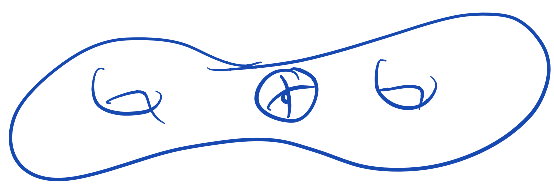


locally symmetric

(4) $T^4 = \mathbb{R}^4/\mathbb{Z}^4$



(5) $\mathbb{H}^2/\Gamma$



$\Gamma < \mathrm{Isom}_+(\mathbb{H}^2)$ discrete & torsion free

Rmk: Ex (4), (5) are "universal examples":

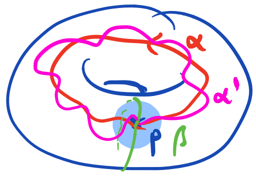
Thm II.4 (Helgason IV.5.6)

A complete simply conn. Riem. locally symm. space is Riem. globally symm. In particular the univ. cov. of a locally symm. space is globally symm. and every loc. symm. space is a quotient of a globally symm. space by a discrete torsion-free gp of isometrie isom. to the fund. gp.

Quick Interlude: Covering theory

$\alpha \simeq \alpha'$ homotopic ⇝ eq. rel.

Ex: $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$



$\pi_1(\mathbb{T}^2, p) = \{ [\gamma]$ equiv. cl. of loops at $\}$

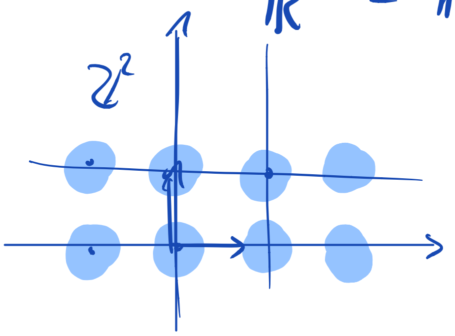
↑

fundamental group

group operation = concatenation

π

$\mathbb{R}^2 = \tilde{\mathbb{T}}^2$ universal covering

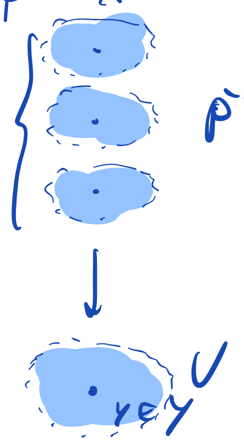


$\pi_1(\tilde{\mathbb{T}}^2) = \{1\}$ ⇝ simply connected.

Covering map $p: X \to Y$

looks locally

sheets



$p^{-1}(V) = \bigsqcup U_i$

Neat properties of symmetric spaces:

Proposition II.7 M Riem. symm. space

Then M is complete (as metric space).
Moreover $\mathrm{Iso}(M)^{\circ}$ acts transitively.

Thm II.10 M RSS. Then $G := \mathrm{Iso}(M)$
has a Lie gp. structure compatible
with the cpt. open top. and it
acts smoothly on M. Moreover

$\mathrm{Iso}(M)/K \longrightarrow M$ is a diffeo

and K contains no non-trivial
normal subgps of G.

$K = G_p = \{ g \in G \mid gp = p \}$.

II) Homogeneous Spaces

Def: A smooth manifold M is called a homogeneous (G-) space / -manifold if there is a Lie group G that acts smoothly and transitively on M.

Ex:

1) $M = \mathbb{R}^n$: $\mathbb{R}^n \curvearrowright \mathbb{R}^n$ via translation;
2) $M = \mathbb{R}^n$: $SL_n\mathbb{R} \ltimes \mathbb{R}^n \curvearrowright \mathbb{R}^n$ via affine trafo's.
$$(A, b) \cdot x = A \cdot x + b.$$
3) $M = S^n$, $O_{n+1}(\mathbb{R}) \curvearrowright S^n$.
4) $M = T^n = \mathbb{R}^n/\mathbb{Z}^n$, $\mathbb{R}^n \curvearrowright \mathbb{R}^n/\mathbb{Z}^n$
5) $M = T^n$, $SL_n\mathbb{Z} \ltimes \mathbb{R}^n \curvearrowright T^n$ via induced affine trafo's: $(A,b)\cdot(x+\mathbb{Z}^n) := Ax+b+\mathbb{Z}^n$
6) Every symmetric space.

Rmk: (i) 1), 2) & 4), 5) show that the group G is usually not unique.
In fact, if $G \curvearrowright M$ transitively and we let another Lie grp G' act trivially on M then $G \times G'$ acts transitively on M: $(g, g') \cdot p = g \cdot p$.

(ii) are all isometric actions.

Characterization of homogeneous spaces:

Thm: Let G be a Lie group, let M be a smooth homogeneous G-manifold, and let $p \in M$. Then the stabilizer group / isotropy group

$$G_p := \{ g \in G \mid gp = p \} \leq G$$

is a closed subgp of G, and the map

$$F : \underset{G \circlearrowright}{G/G_p} \longrightarrow \overset{\circlearrowleft G}{M}, \quad gG_p \longmapsto g \cdot p$$

is a G-equivariant diffeomorphism, i.e.

$$F(g \cdot x) = g \cdot F(x) \quad \forall g \in G \ \forall x \in G/G_p .$$

Rmk: Here G/G_p is equipped with the unique smooth structure s.t. $\pi : G \longrightarrow G/G_p$ is a smooth submersion.

Sketch of Proof:

- G_p is closed: The map $\varphi_p : G \to M$, $g \mapsto g \cdot p$ is cont., whence $G_p = \varphi_p^{-1}(\{p\})$ is closed.

- F is equiv. diffeo: From set-theoretic group actions we know that $F : G/G_p \to M$ is a bijection (orbit-stabilizer thm)

Moreover, it's G-equivariant by defn:

$$F(g \cdot g_0 G_p) = g g_0 \cdot p = g \cdot F(g_0 G_p)$$

Also:

$$\begin{tikzcd} G \arrow[d, "\pi"'] \arrow[dr, bend left, "\tilde{F}"] & \\ G/G_p \arrow[r, "F"] & M \end{tikzcd}$$

$\tilde{F}(g) = g \cdot p$ smooth!

π is a submersion: it admits smooth local sections σ

Thus locally, $F = \tilde{F} \circ \sigma \implies F$ is smooth

F has constant rank, because $G \curvearrowright M$, $G \curvearrowright G/G_p$ are transitive & F is equivariant.

$\implies F$ is a diffeomorphism. $\square$

Q1: When does a homogeneous space $M \cong G/H$ admit a G-inv. Riemannian metric?

Let $\mathcal{R}(M)$ denote the set of Riem. metrics on M.

The group G acts on $\mathcal{R}(M)$ via pull-back:

$m \in \mathcal{R}(M)$, $g \in G$, $p \in M$, $v, w \in T_pM$:

$$(g^* m)_p(v, w) = m_{g \cdot p}(dg_p(v), dg_p(w))$$

Define $\mathcal{R}(M)^G = \{ m \in \mathcal{R}(M) \mid g^* m = m \ \forall g \in G \}$.

Thus Q1 is equivalent to:

<u>Q2</u>: When is $\mathcal{R}(M)^G \neq \emptyset$.

Because M "looks everywhere the same", this question can be further reduced to the "question at a point $p \in M$".

<u>Notation</u>: For a (f.d. real) vector space V

$$\mathrm{Sym}_+(V) := \{ \langle .,. \rangle \text{ inner products on } V \}.$$

<u>Lemma</u>:
$$\varphi : \mathcal{R}(M)^G \longrightarrow \mathrm{Sym}_+(T_pM)^H = \{ H\text{-inv. inner prod. on } T_pM \}$$
$$m \longmapsto m_p$$

is a bijection.

<u>Pf</u>: Clearly, φ is well-def.
Its inverse is given by "translating a given $\langle .,. \rangle \in \mathrm{Sym}_+(T_pM)^H$ around":

$$\varphi^{-1}(\langle .,. \rangle)_q (v, w) = \langle dg_q(v), dg_q(w) \rangle$$

where $g \in G$ s.t. $gp = q$.

This does not depend on the choice of g.

Let $\tilde{g} \in G$ s.t. $\tilde{g} \cdot q = p$. Then $h = \tilde{g} \cdot g^{-1} \in H = G_p$

and $\langle dg_q(v), dg_q(w) \rangle \overset{H\text{-inv.}}{=} \langle dh_p\, dg_q(v), dh_p\, dg_q(w) \rangle$

$$= \langle d\tilde{g}_q(v), d\tilde{g}_q(w) \rangle. \qquad \square$$

Where I planned to arrive:

<u>Prop:</u> The map

$$\mathcal{R}(M)^G \xrightarrow{\varphi} \mathrm{Sym}_+(T_pM)^H \xrightarrow{dF^*} \mathrm{Sym}_+(\mathfrak{g}/\mathfrak{h})^H$$

is a bijection.

$H \curvearrowright \mathrm{Sym}_+(\mathfrak{g}/\mathfrak{h})$

via the adjoint representation

$$\begin{array}{c} Ad : H < G \longrightarrow GL(\mathfrak{g}) \\ \rightsquigarrow \\ H \longrightarrow GL(\mathfrak{g}/\mathfrak{h}) \end{array}$$

$\mathrm{Lie}(G) = \mathfrak{g}$, $\mathrm{Lie}(H) = \mathfrak{h}$.

$\rightsquigarrow$ Use this to show that in Ex 2) and 5) from before there is no invariant Riem. metric.