Lecture

10 March 2021

Last time: affine or C^∞ connection for all smooth mfds.

Defn. M Riem. mfd; a Riemannian conn. on (M,g) is an affine connection that satisfies:

(4) $\nabla_X Y - \nabla_Y X = [X,Y]$

(5) $X\, g(Y,Y') = g(\nabla_X Y, Y') + g(Y, \nabla_X Y')$

Rk If $\gamma: I \to M$ is a smooth curve then (5) can be rewritten as

$$\frac{d}{dt}\, g(Y(t), Y'(t))_{\gamma(t)} =$$

$$= g(\nabla_{\dot\gamma} Y, Y')_{\gamma(t)} + g(Y, \nabla_{\dot\gamma} Y')_{\gamma(t)}$$

In particular if Y, Y' are parallel v.f. along γ $\Rightarrow$ $\nabla_{\dot\gamma} Y = \nabla_{\dot\gamma} Y' = 0$

$\Rightarrow$ $g(Y(t), Y'(t))$ is constant w.r.t. t. $\Rightarrow$ parallel transport preserves the inner product.

Thm (Fund. thm. in Riem. geom.)
Given a Riem. mfd (M,g) $\exists!$ Riem. connection, called the Levi-Civita connection.

Recall (1) $f: M \to M$ diffeo induces a linear map on v.f. via the pushforward

$$(f_* X)_p := d_{f^{-1}(p)} f \; X_{f^{-1}(p)}$$

(2) $\mathrm{Vect}(M) = \mathrm{Der}(C^\infty(M))$ so that $f_* X$ as a derivation is

$$(f_* X)(\varphi) = X(\varphi \circ f^{-1}), \quad \varphi \in C^\infty(M)$$

$\Rightarrow$ f_* is a Lie algebra homo

$$f_*([X,Y]) = [f_* X, f_* Y]$$

Lemma II.13 (M,g) Riem. mfd. with Levi-Civita conn. ∇. Let $\gamma: \mathbb{R} \to M$ smooth

$Y \in \mathrm{Vect}(\gamma^* TM)$ a parallel v.f.
If $f \in \mathrm{Iso}(M)$ $\Rightarrow$ $f_* Y$ is a parallel v.f. along $f \circ \gamma$.

Pf $D: \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$

$$(X,Y) \longmapsto f_*^{-1}(\nabla_{f_* X} f_* Y) = D_X Y$$

To show: D satisfies (1) $\div$ (5)
By uniqueness

$$\nabla_X Y = f_*^{-1}(\nabla_{f_* X} f_* Y) \Rightarrow$$

$$\Rightarrow\; f_*(\nabla_X Y) = \nabla_{f_* X} f_* Y$$

If $X = \dot\gamma$

$$f_* (\underbrace{\nabla_{\dot\gamma} Y}_{=0}) = \nabla_{f_* \dot\gamma} f_* Y$$

$\Rightarrow$ $f_* Y$ is $\parallel$ along $f_* \gamma = f \circ \gamma$

(1) $\div$ (3) are obvious

(4) $D_X Y - D_Y X =$

$$= f_*^{-1}(\nabla_{f_* X} f_* Y) - f_*^{-1}(\nabla_{f_* Y} f_* X)$$

$$= f_*^{-1}(\nabla_{f_* X} f_* Y - \nabla_{f_* Y} f_* X)$$

$$\overset{(4)\text{ for } \nabla}{=} f_*^{-1}([f_* X, f_* Y]) \overset{f_* \text{ Lie alg. homo}}{=} [X,Y].$$

(5) $g(D_X Y, Y') + g(Y, D_X Y') =$

$$= g(f_*^{-1} \nabla_{f_* X} f_* Y, Y') + g(Y, f_*^{-1} \nabla_{f_* X} f_* Y')$$

$$= g(\nabla_{f_* X} f_* Y, f_* Y') + g(f_* Y, \nabla_{f_* X} f_* Y')$$

$$= (f_* X)\, g(f_* Y, f_* Y')$$

$$= X\, g(Y, Y') \quad \blacksquare$$

Rk $\mathrm{Diff}(M) \curvearrowright \mathrm{AffConn}(M)$

$f: M \to M$ diffeo

$\nabla : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ conn.

$\Rightarrow D : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$

$$D_X Y := f_*^{-1}\left(\nabla_{f_* X} f_* Y\right)$$

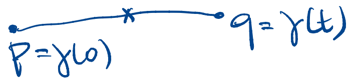
is also an affine conn.

In particular if $M = G$ and $f = L_g \Rightarrow$ we say that ∇ is left invariant if

$$\nabla_X Y = (L_g)^{-1}_* \left(\nabla_{(L_g)_* X} (L_g)_* Y\right). \quad \boxtimes$$

<u>Back to symm. spaces.</u>

$\mathrm{Iso}(M)^\circ \curvearrowright M$ transitively



$\Rightarrow q = s_{\gamma(t/2)} \circ s_p(p)$ are the geod. symm. acting trans. on M.

Defn. The isometries

$$\tau_t := s_{\gamma(t/2)} \circ s_{\gamma(0)}$$

are called *transvections*.

Proposition II.14 M RSS, $\gamma : \mathbb{R} \to M$ a geodesic and τ_t the associated transvection.

(1) $\forall c \in \mathbb{R}$ $\tau_t(\gamma(c)) = \gamma(t+c)$

(2) $d_{\gamma(0)} \tau_t : T_{\gamma(0)} M \to T_{\gamma(t)} M$ is the parallel translation along γ that is if $v = X_{\gamma(0)} \in T_{\gamma(0)} M \Rightarrow$
$(d_{\gamma(0)} \tau_t)\, v = (X^v)_{\gamma(t)}$

(3) The map $t \longmapsto \tau_t$ is a 1-parameter group in $\mathrm{Iso}(M)^\circ$

(4) τ_t is indep. on the param. of γ.

Proof (1) Geod. symm. preserve geod. & reverse orientation. $\Rightarrow \tau_t$ preserves geod. & orientation
$\tau_t(\gamma(c)) = \gamma(c + \text{constant})$. Since
$\tau_t(\gamma(0)) = \gamma(t) \Rightarrow \tau_t(\gamma(c)) = \gamma(c+t)$

(2) We want to show that
$$(d_{\gamma(0)} \tau_t)(X^v)_{\gamma(0)} = (X^v)_{\gamma(t)}$$
and we'll show that
$$(d_{\gamma(c)} \tau_t)(X^v)_{\gamma(c)} = (X^v)_{\gamma(t)}$$

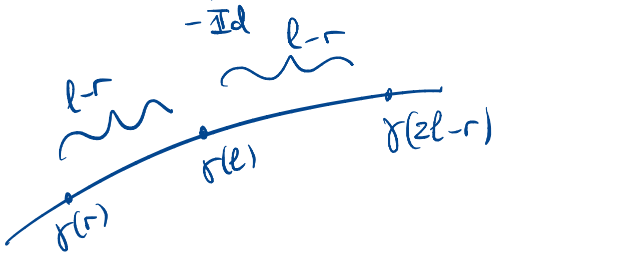
$$(d_{\gamma(c)} \tau_t)(X^v)_{\gamma(c)} =$$
$$= d_{\gamma(c)}\left(s_{\gamma(t/2)} \circ s_{\gamma(0)}\right)(X^v)_{\gamma(c)} =$$
$$= \left(d_{\gamma(-c)} s_{\gamma(t/2)}\right)\left(d_{\gamma(c)} s_{\gamma(0)}\right)(X^v)_{\gamma(c)}$$

$\Rightarrow$ we are going to compute
$(d_{\gamma(r)} s_{\gamma(\ell)})(X^v)_{\gamma(r)}$ for general r, ℓ.

using that X^v is parallel along γ

$\overset{\text{II.13}}{\Rightarrow} (s_{\gamma(\ell)})_* X^v$ is parallel along $s_{\gamma(\ell)} \circ \gamma = \gamma$

$$(s_{\gamma(\ell)})_* X^v_{\gamma(\ell)} = d_{s^{-1}_{\gamma(\ell)}(\gamma(\ell))} s_{\gamma(\ell)}\, X^v_{s^{-1}_{\gamma(\ell)}(\gamma(\ell))}$$
$$= \underbrace{\left(d_{\gamma(\ell)} s_{\gamma(\ell)}\right)}_{= -\mathrm{Id}} X^v_{\gamma(\ell)} = -X^v_{\gamma(\ell)}$$



$$s_{\gamma(\ell)}\, \gamma(r) = \gamma(2\ell - r)$$

$$-X^\nu_{\gamma(2l-r)} = (s_{\gamma(l)})_* X^\nu_{\gamma(2l-r)} =$$

$$= d_{s^{-1}_{\gamma(l)}(\gamma(2l-r))} s_{\gamma(l)} X^\nu_{s^{-1}_{\gamma(l)}(\gamma(2l-r))}$$

$$= (d_{\gamma(r)} s_{\gamma(l)}) X^\nu_{\gamma(r)}$$

$$\Rightarrow (d_{\gamma(r)} s_{\gamma(l)}) X^\nu_{\gamma(r)} = -X^\nu_{\gamma(2l-r)}$$

$l=0 \quad r=c \Rightarrow (d_{\gamma(c)} s_{\gamma(0)}) X^\nu_{\gamma(c)} = -X^\nu_{\gamma(-c)}$

$l=t/2, \; r=-c \Rightarrow d_{\gamma(-c)} s_{\gamma(t/2)} X^\nu_{\gamma(-c)} = -X^\nu_{\gamma(t+c)}$

$\Rightarrow$

$$d_{\gamma(c)}\tau_t X^\nu_{\gamma(c)} = \underbrace{(d_{\gamma(-c)} s_{\gamma(t/2)}) \underbrace{(d_{\gamma(c)} s_{\gamma(0)})(X^\nu)_{\gamma(c)}}_{= -X^\nu_{\gamma(-c)}}}_{= +X^\nu_{\gamma(t+c)}}$$

$$\Rightarrow d_{\gamma(c)}\tau_t = P_{\gamma,[c,t+c]}.$$

(3) $\tau_{t_1+t_2} = \tau_{t_2}\circ\tau_{t_1}$

$$d_{\gamma(c)}\tau_{t_1+t_2} = P_{\gamma,[c,t_1+t_2+c]} =$$

$$= P_{\gamma,[c+t_1,c+t_1+t_2]}\circ P_{\gamma,[c,c+t_1]} =$$

$$= (d_{\gamma(c+t_1)}\tau_{t_2})(d_{\gamma(c)}\tau_{t_1}) =$$

$$= d_{\gamma(c)}(\tau_{t_2}\circ\tau_{t_1})$$

Also $\tau_{t_1+t_2}(\gamma(c)) = (\tau_{t_2}\circ\tau_{t_1})(\gamma(c))$

$\Rightarrow \tau_{t_1+t_2} = \tau_{t_2}\circ\tau_{t_1}$ (lemma II.2?)

(4) A unit speed repar. of γ is $t\mapsto t+a$. Thus

$$s_{\gamma(t/2+a)} s_{\gamma(a)} = \underbrace{s_{\gamma(t/2+a)} s_{\gamma(0)}} s_{\gamma(0)} s_{\gamma(a)}$$

$$= \tau_{t+2a}(s_{\gamma(a)} s_{\gamma(0)})^{-1} =$$

$$= \tau_{t+2a}(\tau_{2a})^{-1} \overset{(3)}{=} \tau_t. \qquad ∎$$

Defn. The map $\mathbb{R}\to \mathrm{Iso}(M)^0$, $t\mapsto\tau_t$ is called one-par. gp of transvections associated to the geodesic γ.

Algebraic point of view of RSS

$\rightsquigarrow$ 1-par. gp in $\mathrm{Iso}(M)^0$

$\Updownarrow$

line in $\mathrm{Lie}(\mathrm{Iso}(M)^0)$

M RSS $\Rightarrow M = G/K$, K cpt, $G = \mathrm{Iso}(M)$

Q: Which pairs (G,K) give rise to a RSS?

Defn A Lie gp autom. $\sigma: G\to G$ is an involution if $\sigma^2=\mathrm{id}_G$ and $\sigma\neq \mathrm{id}_G$.

Proposition II.15 M RSS and $o\in M$ a basept, $G = \mathrm{Iso}(M)^0$ and $K = \mathrm{Stab}_G(o)$. Then the autom

$$\sigma: G\to G, \quad g\mapsto s_o g s_o$$

is an involution of G and $(G^\sigma)^0 \subseteq K \subseteq G^\sigma$

Pf $s_o = s_o^{-1} \Rightarrow \sigma(g) := s_o g s_o^{-1}$ is an autom. of G.

$$\sigma^2(g) = s_o(s_o g s_o)s_o = s_o^2 g s_o^2 = g$$

$K\subseteq G^\sigma$: To show $\sigma(k) = s_o k s_o = k$

$$\sigma(k)(o) = s_o k s_o(o) = s_o k.o = s_o(o) = o = k.o$$

$$d_o\sigma(k) = d_o(s_o k s_o) = \underbrace{(d_o s_o)}_{-Id}(d_o k)\underbrace{(d_o s_o)}_{-Id} =$$

$$= d_o k \Rightarrow \sigma(k)=k \text{ (lemma II.2)}$$

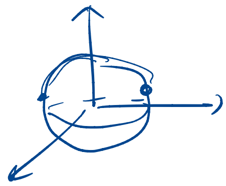
$(G^\sigma)^0\subseteq K$: It will be enough to show that a nbd. of $e\in G^\sigma$ is contained in K.

Let $V \subset M$ be an open nbd of $o \in M$.
By cont. of G-action on M
$\exists$ $U \subset G^\sigma$, $U \ni e$ s.t. $g.o \in V$
$\forall g \in U$. Want to show that $U \subset K$,
that is $g.o = o$ $\forall g \in U$.
In fact, since $g \in G^\sigma \Rightarrow \sigma(g) = g$
$\Rightarrow s_o g s_o = g \Rightarrow g.o = s_o g s_o(o) =$
$= s_o(g.o) \Rightarrow g.o$ is a fixed pt
of s_o. Since s_o has isolated
fixed pt and o is a fixed pt,
by restricting U if necessary,
$g.o = o$. $\boxed{}$

Rk Cannot be more precise about K.

Ex $M = S^2$, $G = \mathrm{Iso}(M)^0 = SO(3)$
$o = e_3 \in S^2$

$\Rightarrow s_o = \begin{pmatrix} -Id_2 & 0 \\ 0 & 1 \end{pmatrix}$, $g = \begin{pmatrix} A & b \\ c & d \end{pmatrix}$



$\sigma(g) = s_o g s_o = \begin{pmatrix} A & -b \\ -c & d \end{pmatrix}$

$\Rightarrow G^\sigma = \left\{ g \in SO(3) : g = \begin{pmatrix} A & 0 \\ 0 & d \end{pmatrix}, \begin{matrix} A \in O(2) \\ \det A \cdot d = 1 \end{matrix} \right\}$
has two conn. cpt.

connected
$S^2 = SO(3)/K$ simply connected $\Rightarrow$
$\Rightarrow K$ connected $\Rightarrow (G^\sigma)^0 = K =$
$= \left\{ g \in SO(3) : g = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, A \in SO(2) \right\}$.

Ex $M = \mathbb{P}(\mathbb{R}^3) \cong S^2/\pm Id$
$G = \mathrm{Iso}(M)^0 = O(3,\mathbb{R})/\pm Id$, $o = [e_3]$
$\Rightarrow (G^\sigma)^0 = K = G^\sigma$.