# Lecture

11 March 2021

Yesterday: $M$ RSS, $o \in M$, $K = \mathrm{Stab}_G(o)$, $G = \mathrm{Iso}(M)^\circ$

$\Rightarrow$ $\sigma : G \to G$, $g \mapsto s_o g s_o$ is an involution and $(G^\sigma)^\circ \leq K \leq G^\sigma$

Today: • look at the converse of this statement

• Cartan decomposition

**Recall** $\forall g \in G$, $c_g(h) := g h g^{-1}$ Lie gp. auto $\Rightarrow$ if $\mathfrak{g} = \mathrm{Lie}(G)$

$\Rightarrow$ $d_e c_g : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homo, called the Adjoint representation of $G$

$\mathrm{Ad} : G \to GL(\mathfrak{g})$

$\mathrm{Ad}_G(g) := d_e c_g$

symmetry at $o \in M$, then

$$s_o \circ \pi = \pi \circ \sigma$$

**Corollary II.17** The geodesic sym. $s_o$ is indep. of the choice of Riem. metric on $M$.

**Rk** Recall that

$$K / K \cap Z(G) \xrightarrow{\cong} \mathrm{Ad}_G(K) \subset GL(\mathfrak{g})$$

Let $G = \widetilde{SL(2,\mathbb{R})}$, $\sigma : SL(2,\mathbb{R})$ ↺

$\sigma(g) := {}^t g^{-1}$. Let $\tilde{\sigma} : G \to G$ to be the unique lift of $\sigma$. Let $p : G \to SL(2,\mathbb{R})$ be the covering map $\Rightarrow$

$$G^\sigma = p^{-1}(SO(2,\mathbb{R})) \simeq \mathbb{R}.$$

Thus $G^\sigma$ is not cpt but

$$\mathrm{Ad}_G(G^\sigma) \cong SO(2)/\pm \mathrm{Id}$$ is compact.

**Definition** Let $G$ be a connected Lie group and $K \leq G$ a closed sbgp. The pair $(G,K)$ is a **Riemannian symmetric pair** if:

(i) $\mathrm{Ad}_G(K)$ is a compact sbgp of $GL(\mathfrak{g})$

(ii) $\exists$ an involutive auto $\sigma \in \mathrm{Aut}(G)$ such that $(G^\sigma)^\circ \leq K \leq G^\sigma$

Yesterday RSS $\Rightarrow$ RSP

Today RSP $\Rightarrow$ RSS

**Theorem II.16** Let $(G,K)$ be a RSP with involutive auto $\sigma$. Then $G/K$ is a RSS with respect to any $G$-invariant Riemannian metric. If $\pi : G \to G/K$ is the projection and $s_o$ is the geodesic

**Examples** (1) $G < GL(n,\mathbb{R})$ closed conn. under transposition (e.g. $SL(n,\mathbb{R})$, $Sp(2n,\mathbb{R})$, $SO(p,q)^\circ$). Let $\sigma \in \mathrm{Aut}(G)$ $\sigma(g) := {}^t g^{-1}$. If $G \not\subset O(n,\mathbb{R})$ then $\sigma$ is involutive.

$G^\sigma = G \cap O(n,\mathbb{R})$, and it is a fact that $G^\sigma$ is connected so that $(G, G^\sigma)$ is a RSP.

($G^\sigma$ is connected since $G/G^\sigma$ is simply connected)

(2) $G \leq GL(n,\mathbb{C})$ closed conn. sbgp. invariant under $g \overset{\sigma}{\mapsto} g^* = {}^t\bar{g}^{-1}$. If $G \not\subset U(n)$, then $\sigma$ is an invol. and $G^\sigma := G \cap U(n)$ is connected. $\Rightarrow (G, G^\sigma)$ is a RSP, where for ex. $G = SL(n,\mathbb{C})$, $GL(n,\mathbb{C})$, $Sp(2n,\mathbb{C})$, $SO(n,\mathbb{C})$.

(3) $G = SO(n,\mathbb{R})$, $\mathbb{R}^n = \mathbb{R}^p \oplus \mathbb{R}^q$,

$r \in SO(n,\mathbb{R})$ : $r|_{\mathbb{R}^p} = Id_p$ & $r|_{\mathbb{R}^q} = -Id_q$ $\quad r = \begin{pmatrix} Id_p & 0 \\ 0 & -Id_q \end{pmatrix}$

Then $G^\sigma = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : \begin{array}{l} A \in O(p) \\ B \in O(q) \\ \det A \det B = 1 \end{array} \right\}$

has two connected comp.

and $K = (G^\sigma)^0$ or $K = G^\sigma$.

For ex. if $p = 1$

- $K = (G^\sigma)^0 \Rightarrow G/K \simeq SO(n)/K \simeq S^{n-1}$
- $K = G^\sigma \Rightarrow G/K = \mathbb{P}(\mathbb{R}^n)$

(4) Argue similarly for $U(n)$.

**Recall** $G$ Lie gp, $\mathfrak{g} = Lie(G)$

$\exp : \mathfrak{g} \to G$ defined as

$\exp(X) := \varphi_X(1)$, where

$\varphi_X : \mathbb{R} \to G$ is the unique 1-parameter sbgp of $G$ s.t.

$\varphi_X(0) = e$, $\dot{\varphi}_X(0) = X$.

If $h : G \to G$ is a homo

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{d_e h} & \mathfrak{g} \\ {\scriptstyle \exp}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{h} & G \end{array} \qquad h(\exp X) = \exp(d_e h X)$$

In particular if $h = c_g \Rightarrow$

$\Rightarrow g(\exp X)g^{-1} = \exp(Ad_G(X))$.

**Lemma II.18** $(G,K)$ RSP, $\mathfrak{g} = Lie(G)$, $\mathfrak{k} := Lie(K)$. Then

(i) $\mathfrak{k} = \{X \in \mathfrak{g} : d_e\sigma X = X\}$ and

(ii) if $\mathfrak{p} := \{X \in \mathfrak{g} : d_e\sigma X = -X\}$,

then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Pf $(G^\sigma)^0 \subseteq K \subseteq G^\sigma$

$\Rightarrow \mathfrak{k} = Lie(G^\sigma) =$

$= \{X \in \mathfrak{g} : \exp(tX) \in G^\sigma \ \forall t \in \mathbb{R}\}$

$= \{X \in \mathfrak{g} : \sigma(\exp tX) = \exp tX \ \forall t \in \mathbb{R}\}$

$\overset{(*)}{=} \{X \in \mathfrak{g} : \exp(d_e\sigma(tX)) = \exp(tX) \ \forall t \in \mathbb{R}\}$

$= \{X \in \mathfrak{g} : d_e\sigma(X) = X\}$

For (ii) $\Rightarrow (d_e\sigma)^2 = d_e\sigma^2 = Id_{\mathfrak{g}}$

$\Rightarrow d_e\sigma$ is diagon. with e.v. $\pm 1$

$$X = \underbrace{\tfrac{1}{2}(X + d_e\sigma(X))}_{\in \mathfrak{k}} + \underbrace{\tfrac{1}{2}(X - d_e\sigma(X))}_{\in \mathfrak{p}}$$

**Lemma II.19** $(G,K)$ RSP with $\sigma$ and $\mathfrak{p} = \{X \in \mathfrak{g} : d_e\sigma(X) = -X\}$.

Then $\mathfrak{p}$ is $Ad_G(K)$-invariant.

**Pf** Notice that $\sigma \circ c_k = c_k \circ \sigma$

In fact

$\sigma \circ c_k(g) = \sigma(kgk^{-1}) =$

$= \sigma(k)\sigma(g)\sigma(k)^{-1} \underset{k \subset G^\sigma}{=} k\sigma(g)k^{-1} =$

$= c_k \circ \sigma(g)$.

Differentiate at $e$.

$d_e(\sigma \circ c_k) = (d_e\sigma)(d_e c_k)$

$\shortparallel$

$d_e(c_k \circ \sigma) = (d_e c_k)(d_e\sigma)$

$\Rightarrow d_e\sigma(Ad(k)) \overset{(**)}{=} Ad(k)\, d_e\sigma$

If $X \in \mathfrak{p} \Rightarrow d_e\sigma(X) = -X$

Apply (**) to $X \in \mathfrak{p} \Rightarrow$

$\Rightarrow Ad_G(k)X \in \mathfrak{p}$. ■

**Proof of Thm II.16**

First of all the diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{c_k} & G \\ {\scriptstyle \pi}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \pi} \\ G/K & \xrightarrow{\tau_k} & G/K \end{array}$$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "Ad_G(k)"] \arrow[d, "d_e\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{g} \arrow[d, "d_e\pi"] \\ T_o G/K \arrow[r, "d_o k"'] & T_o G/K \end{tikzcd}$$

where $o = \pi(e)$

and the diagram of the differentials commutes as well.

$$d_e\pi \circ Ad(k) = d_o k \circ d_e\pi$$

Moreover $d_e\pi$ is surjective and has kernel $\mathfrak{k} = \ker(d_e\pi)$

$$\Rightarrow \begin{tikzcd} \mathfrak{p} \arrow[r, "Ad_G(k)"] \arrow[d, "d_e\pi"'] \arrow[dr, phantom, "\circlearrowright"] & \mathfrak{p} \arrow[d, "d_e\pi"] \\ T_o G/K \arrow[r, "d_o k"'] & T_o G/K \end{tikzcd}$$

$\Rightarrow \mathfrak{p} \simeq T_o\, G/K$ as vector spaces

and as $K$-spaces, where the action of $K$ on $\mathfrak{p}$ is via $Ad_G$ and on $T_o G/K$ is given by $d_o k$.

Since $Ad_G(K)$ is compact, if $B' : \mathfrak{p} \times \mathfrak{p} \to \mathbb{R}$ is a positive definite inner product and $\mu$ is the Haar m on $Ad_G(K)$, then $\forall\, X, Y \in \mathfrak{p}$

$$B(X,Y) := \int_{Ad_G(K)} B'(k_* X, k_* Y)\, d\mu(k)$$

is an $Ad_G(K)$-inv. inner product that can be shown to be $\neq 0$.

Set $Q_o : T_o G/K \times T_o G/K \to \mathbb{R}$ to be $Q_o(X,Y) := B((d_e\pi)^{-1}(X), (d_e\pi)^{-1}(Y))$

If $X_p, Y_p \in T_p G/K$, define

$$Q_p(X_p, Y_p) := Q_o(d_o g^{-1} X_p, d_o g^{-1} Y_p),$$

where $go = p$. $Q_p$ is well-defined since $Q_o$ is $K$-invariant. If $go = ho$

$$Q_o(d_o g^{-1} X_p, d_o g^{-1} Y_p) \overset{K\text{-inv.}}{=}$$
$$= Q_o(d_o(h^{-1}g)\, d_o g^{-1} X_p, d_o(h^{-1}g)\, d_o g^{-1} X_p)$$
$$= Q_o(d_o h^{-1} X_p, d_o h^{-1} Y_p)$$

$\Rightarrow$ Have a $G$-inv. Riem. metric on $G/K$.

Can now define $s_o \in Iso(M)$

Want to have $s_o \circ \pi = \pi \circ \sigma$

$( s_o(gK) = \sigma(g)K,\ s_o(g \cdot o) = \sigma(g) \cdot o )$.

Then define $s_o := \pi \circ \sigma \circ \pi^{-1}$.

Since $G/K$ is homogeneous and $s_{g \cdot o} = g s_o g^{-1}$ (proof of lemma I.7.3)

$\Rightarrow$ get $s_p\ \forall\, p \in M$.

We see first that $s_o$ is well-dfn.

$$s_o(x) = \pi \circ \sigma \circ \pi^{-1}(x) = \pi \circ \sigma(xK) =$$
$$= \pi(\sigma(x)\sigma(K)) = \pi(\sigma(x))K =$$
$$= \pi(\sigma(x)).$$

Now we show that $s_o^2 = Id$.

$$s_o^2 \circ \pi = s_o \circ (s_o \circ \pi) = s_o \circ (\pi \circ \sigma) =$$
$$= (s_o \circ \pi) \circ \sigma = (\pi \circ \sigma) \circ \sigma =$$
$$= \pi \circ \sigma^2 = \pi.$$

Next time: $d_o s_o = -Id$

$s_o$ is an isometry