lecture

17 March 2021

(G,K) Riem. symm. pair if G, K are Lie gps $K \le G$ closed &

(i) $\exists\ \sigma \in \mathrm{Aut}(G)$ involution s.t. $(G^\sigma)^\circ \le K \subseteq G^\sigma$

(ii) $\mathrm{Ad}_G(K)$ is cpt.

Thm II.16 (G,K) RSP with σ $\Rightarrow$ G/K is a RSS w.r.t. G-inv. Riem. metric. If $\pi : G \to G/K$ is the proj. $\Rightarrow$ $s_o \circ \pi = \pi \circ \sigma$, $o \in G/K$ basept.

Pf (1) $o \in G/K$ basept, $K = \mathrm{Stab}_G(o)$ $\exists$ iso $\mathfrak{p} \xrightarrow{d_e\pi} T_o G/K$, where $\mathfrak{p}$ = eigenspace of e. value -1 of $d_e\sigma$.

Moreover $d_e\pi$ is a K-map, where $K \curvearrowright \mathfrak{p}$ via Ad_G and $K \curvearrowright T_o G/K$ via the diff.

(2) Take on $\mathfrak{p}$ an inner product, make it $\mathrm{Ad}_G(K)$-inv., push it over to $T_o G/K$ and define it $\forall\ p \in G/K$ by G-invariance (possible because K-inv.)

(3) $s_o := \pi \circ \sigma \circ \pi^{-1}$ well-defn. & $s_o^2 = \mathrm{id}$.

(4) $d_p s_p = -\mathrm{id}$ $\forall\ p \in G/K$ | To show

(5) s_o is a Riem. isometry,

(4) $s_o \circ \pi = \pi \circ \sigma$ diff. at e

$$(d_o s_o)(d_e\pi) = (d_e\pi)(d_e\sigma)$$

If $X \in \mathfrak{p}$ $\Rightarrow$

$$(d_o s_o)(d_e\pi)(X) = (d_e\pi)\underbrace{(d_e\sigma)(X)}_{=-X}$$

$\Rightarrow$ $(d_o s_o)\underbrace{(d_e\pi)(X)} = -(d_e\pi)(X)$

$\Rightarrow$ ok for o.

Recall that $s_{go} = g s_o g^{-1}$ $\Rightarrow$

$$\Rightarrow d_{go} s_{go} = d_{go}(g s_o g^{-1}) =$$

$$= (d_o g)\underbrace{(d_o s_o)}_{=-\mathrm{Id}}(d_{go} g^{-1}) =$$

$$= -(d_o g)(d_{go} g^{-1}) = -\mathrm{Id}.$$

chain rule.

(5) To see that s_o is an isometry, that is it preserves any G-inv. Riem. metric Q

$$Q_p(X_p, Y_p) = Q_{s_o(p)}((d_p s_o)X_p, (d_p s_o)Y_p)$$

$\forall\ p \in M$ & $\forall\ X_p, Y_p \in T_pM$.

We write $p = g\cdot o$, $X_o, Y_o \in T_oM$

and $X_p = (d_o g)X_o$, $Y_p = (d_o g)Y_o$ $\Big\}$ $\Rightarrow$

G-inv. $\Rightarrow$ $Q_p(X_p, Y_p) = Q_o(X_o, Y_o)$.

To prove that

$$Q_{s_o(p)}((d_p s_o)X_p, (d_p s_o)Y_p) = Q_o(X_o, Y_o).$$

We saw already that $s_o(go) = \sigma(g)(o)$ but we need that

$$s_o \circ g = \sigma(g) \circ s_o \quad \forall\ xK \in M$$

In fact

$$s_o \circ g(xK) = s_o \circ \pi(gx) =$$
$$= \pi \circ \sigma(gx) = \sigma(gx)K =$$
$$= \sigma(g)\underline{\sigma(x)K} = \sigma(g)\pi\circ\sigma(x)$$
$$= \sigma(g)\, s_o \circ \pi(x) =$$
$$= \sigma(g)\, s_o(xK).$$

$$Q_{s_0(p)}\big((d_p s_0)X_p,(d_p s_0)Y_p\big)=$$
$$=Q_{s_0(p)}\big((d_p s_0)(d_0 g)X_0,(d_p s_0)(d_0 g)Y_0\big)$$
$$=Q_{s_0(p)}\big(d_0(s_0\circ g)X_0,\ d_0(s_0\circ g)Y_0\big)$$
$$=Q_{s_0(p)}\big(d_0(\sigma(g)\circ s_0)X_0,\ d_0(\sigma(g)\circ s_0)Y_0\big)$$
$$=Q_{s_0(p)}\big((d_0\sigma(g))(d_0 s_0)X_0,\ (d_0\sigma(g))(d_0 s_0)Y_0\big)$$
$$=Q_{s_0(p)}\big(d_0\sigma(g)X_0,\ d_0\sigma(g)Y_0\big)$$
$$=Q_0(X_0,Y_0).$$ ∎

Remark M RSS $\rightsquigarrow (G,K)$ associated RSP w.r.t. an involution $\sigma\in \mathrm{Aut}(G)$.
We'll prove that σ is unique.

$\sigma_i\in\mathrm{Aut}(G)$ two inv. of G s.t. $(G^{\sigma_i})^0\subseteq K\subseteq G^{\sigma_i}$, $i=1,2$. Then
$$\pi\circ\sigma_1=s_0\circ\pi=\pi\circ\sigma_2$$
$$\Rightarrow \sigma_1(h)(o)=\sigma_2(h)(o)\quad\forall h\in G$$
Want to see that
$$\sigma_1(h)(p)=\sigma_2(h)(p)\quad\forall p\in M\ \ \forall h\in G.$$
Let $g\in G$ be such that $g.o=p$ and let g' be such that $\sigma_1(g')=g$. $\Rightarrow \sigma_1(g')(o)=\sigma_2(g')(o)$.
But also $\sigma_1(hg')(o)=\sigma_2(hg')(o)$ $\forall h\in G$

$\Rightarrow \sigma_1(h)p=\sigma_2(h)p.$ ∎

Defn. Let (G,K) be a RSP with inv. σ. The **Cartan involution** of $\mathfrak{g}$ is the autom. $\Theta:=d_e\sigma:\mathfrak{g}\to\mathfrak{g}$.
The corresp. eigenspace dec. $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is the **Cartan decomposition of $\mathfrak{g}$** with respect to Θ.

Proposition II.20 (G,K) RSP and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is the Cartan decomp. w.r.t. Θ, then
$[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$, $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$

Pf X,Y e.vectors of Θ
$$\Theta X=\lambda X,\quad \Theta Y=\mu Y$$
where $\lambda,\mu\in\{\pm1\}$.
Θ is a Lie alg. homo
$\Rightarrow$
$$\Theta([X,Y])=[\Theta X,\Theta Y]=$$
$$=\lambda\mu[X,Y]$$ ∎

II.1 Isom. & the isom. gp.
II.2 Geod. symm.
II.3 Transv. & parallel tr.
II.4 Algebraic pt. of view

II.5 Exponential maps & geodesics

M RSS, $G=\mathrm{Iso}(M)^0$
$M=G/K$, $o\in M$, $K=\mathrm{Stab}_G(o)$

$\exp : \mathfrak{g} \to G$ Lie gp. exp. map.

$\mathrm{Exp}_o : T_oM \to M$ Riem. exp. map

Thm II.21 $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan dec. of $\mathfrak{g}$. Then the diagram

$$\begin{tikzcd}
\mathfrak{p} \arrow[r, "d_e\pi"] \arrow[d, "\exp"'] & T_oM \arrow[d, "\mathrm{Exp}_o"] \\
G \arrow[r, "\pi"'] & M
\end{tikzcd}$$

commutes. In particular if $X \in \mathfrak{p}$ then

$$t \longmapsto (\exp tX)_*(o) \in M$$

is the geodesic through $o \in M$ at $t=0$ with tg vector $d_e\pi(X) \in T_oM$

Pf If $X \in \mathfrak{p}$ let

$\gamma(t) := \mathrm{Exp}_o(t\, d_e\pi(X))$

the geod. in M through o at $t=0$ and with tg vector $d_e\pi(X) \in T_oM$.

Let τ_t be transvection along γ, $\tau_t = s_{\gamma(t/2)} \circ s_{\gamma(0)} =$

$= s_{\gamma(t/2)} \circ s_o$. Since τ_t is a 1-par. subgp in G $\Rightarrow$

$\Rightarrow \exists\, Y \in \mathfrak{g}$ s.t. $\tau_t = \exp(tY) \in G$.

We have two geod.

$\gamma(t) = \mathrm{Exp}_o(t\, d_e\pi(X))$

$t \longmapsto \exp(tY) \in G$

both through o at $t=0$

Moreover

$\pi(\exp tY) = \pi(\tau_t) =$

$= \pi(s_{\gamma(t/2)} \circ s_{\gamma(0)}) =$

$= s_{\gamma(t/2)} \circ s_{\gamma(0)}(o) = s_{\gamma(t/2)}\gamma(0) =$

$= \gamma(t) \Rightarrow$

$\Rightarrow \pi(\exp tY) = \mathrm{Exp}_o(t\, d_e\pi(X))$

If we show that $X = Y$ $\Rightarrow$ done.

To do this, we evaluate the derivative at $t=0$.

By defn. the tg vector at $t=0$ to $\gamma(t)$ is $d_e\pi(X)$

Also

$$\frac{d}{dt}\Big|_{t=0} \pi(\exp tY) = d_e\pi \frac{d}{dt}\Big|_{t=0} (\exp tY)$$

$$= d_e\pi(Y).$$

$\Rightarrow d_e\pi(X) = d_e\pi(Y)$

To conclude that $X = Y$ we need to show that $Y \in \mathfrak{p}$.

To see this we'll show that $d_e\sigma(Y) = -Y$.

Since $\sigma(g) = s_{\gamma(0)}\, g\, s_{\gamma(0)}$

$\exp(t\, d_e\sigma(Y)) \stackrel{?}{=} \exp(-tY)$

$\exp(t\, d_e\sigma(Y)) =$

$= \exp(d_e\sigma(tY)) =$

$= \sigma(\exp(tY)) =$

$= s_{\gamma(0)} \exp(tY)\, s_{\gamma(0)} =$

$= s_{\gamma(0)}\, \tau_t\, s_{\gamma(0)} =$

$= s_{\gamma(0)}\, s_{\gamma(t/2)} \underbrace{s_{\gamma(0)}\, s_{\gamma(0)}}_{= id}$

$$= S_{\gamma(0)} S_{\gamma(t/2)} = S^{-1}_{\gamma(0)} S^{-1}_{\gamma(t/2)} =$$
$$= (S_{\gamma(t/2)} \circ S_{\gamma(0)})^{-1} =$$
$$= (\tau_t)^{-1} = \tau_{-t} = \exp(-tY). \blacksquare$$

Rk Exp does not depend on the Riem. metric on M (RSS).

Now we are going to evaluate the der. of Exp at any pt.

$$\pi(\exp(tX)) = \mathrm{Exp}_o(d_e\pi(tX))$$

Thm II.22 Let G be a Lie gp with $\mathrm{Lie}(G) = \mathfrak{g}$ and $\exp : \mathfrak{g} \to G$ be the exp. map. Let $X \in \mathfrak{g}$. Under the identif. $T_X\mathfrak{g} \simeq \mathfrak{g}$ $\Rightarrow$

$\Rightarrow$ $d_X \exp : \mathfrak{g} \to T_{\exp(X)} G$

is given by

$$d_X \exp = d_e L_{\exp X} \left(\sum_{n=0}^{\infty} \frac{(\mathrm{ad}(X))^n}{(n+1)!} \right)$$

To prove this:

(1) Shlomo Sternberg "Lie algebras"

(2) notes

Recall $\mathrm{Ad}_G : G \to GL(\mathfrak{g})$

Lie gp. homo $\Rightarrow$

$d_e \mathrm{Ad}_G : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

is called the adjoint representation of $\mathfrak{g}$

$\mathrm{ad}_{\mathfrak{g}} : \mathfrak{g} \to GL(\mathfrak{g})$.

By naturality of exp $\Rightarrow$

$$\Rightarrow \mathrm{Ad}_G(\exp tX) = \exp(t\, \mathrm{ad}_{\mathfrak{g}}(X))$$

Using this one can prove that

$$\mathrm{ad}_{\mathfrak{g}}(X)(Y) = [X, Y]$$

$\forall X, Y \in \mathfrak{g}$.

Corollary II.23

Let $\mathrm{Exp}_o \circ d_e\pi : \mathfrak{p} \to M$.

Then if $X \in \mathfrak{p}$ the diff.

$d_X(\mathrm{Exp}_o \circ d_e\pi) : \mathfrak{p} \to T_{(\mathrm{Exp}_o \circ d_e\pi)(X)(o)} M$

is given by

$$d_X(\mathrm{Exp}_o \circ d_e\pi) =$$
$$= (d_o L_{\exp X} \circ d_e\pi) \sum_{n=0}^{\infty} \frac{(T_X)^n}{(2n+1)!}$$

where $T_X = (\mathrm{ad}(X))^2$

Rk

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "L_g"'] & G/K \arrow[d, "L_g"] \\ G \arrow[r, "\pi"] & G/K \end{tikzcd}$$

$X \in \mathfrak{p}$

$$(\mathrm{ad}_{\mathfrak{g}}(X))^{2n+1}(\mathfrak{p}) =$$

$$= \underbrace{(\mathrm{ad}_{\mathfrak{g}}(X))^{2n} \underbrace{\mathrm{ad}_{\mathfrak{g}}(X)(\mathfrak{p})}_{\subset \mathfrak{k}}}_{\subset \mathfrak{k}}$$

$$\mathfrak{k} \subset \ker d_e\pi$$