Symmetric Spaces: Exercise Class 2

18.3.21

I) Recap: Slogan for the lectures so far:

"Develop a dictionary between the geometry of symmetric spaces and the theory of Lie groups."

Riemannian symmetric spaces $\longleftrightarrow$ Riemannian symmetric pairs

Definition Let G be a connected Lie group and $K \leq G$ a closed sbgp. The pair (G, K) is a Riemannian symmetric pair if:

(i) $\mathrm{Ad}_G(K)$ is a compact sbgp of $GL(\mathfrak{g})$

(ii) $\exists$ an involutive auto $\sigma \in \mathrm{Aut}(G)$ such that $(G^\sigma)^\circ \subseteq K \subseteq G^\sigma$

$\boxed{M \text{ RSS}, o \in M, s_o : M \to M \text{ geod. symmetry at } o.}$ $\longmapsto$ $\boxed{G := \mathrm{Iso}(M)^\circ, \; K := \mathrm{stab}_o(G), \; \sigma : G \to G, \; g \mapsto s_o g s_o}$

Prop II.15: (G, K) is a RSP.

$\pi : G \longrightarrow G/K =: M$,
$o = eK \in G/K$,
Def: $s_o : G/K \to G/K$ by
$s_o \circ \pi = \pi \circ \sigma$

G conn. Lie gp, $K \leq G$ closed,
$\sigma : G \to G$ involution s.t.:
(i) $Ad_G(K) \subseteq GL(\mathfrak{g})$ is cpt.
(ii) $(G^\sigma)^0 \subseteq K \subseteq G^\sigma$.

Thm II.16: For every G-inv. Riem. metric M is an RSS with geod. symmetry s_o at $o \in M$.

Riemannian exponential map $Exp : T_oM \to M$ VS Lie theoretic $exp : \mathfrak{g} \to G$

Thm II.21 $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan dec. of $\mathfrak{g}$. Then the diagram

$$\begin{tikzcd}
\mathfrak{p} \arrow[r, "d_e\pi", "\cong"'] \arrow[d, "exp"'] \arrow[dr, phantom, "\circlearrowright"] & T_oM \arrow[d, "Exp_o"] \\
G \arrow[r] & M \\[-20pt]
g \arrow[r, mapsto, "\pi"] & g \cdot o
\end{tikzcd}$$

commutes. In particular if $X \in \mathfrak{p}$ then
$$t \longmapsto (\exp tX)_*(o) \in M$$
is the geodesic through $o \in M$ at $t=0$ with tg vector $d_e\pi(X) \in T_oM$

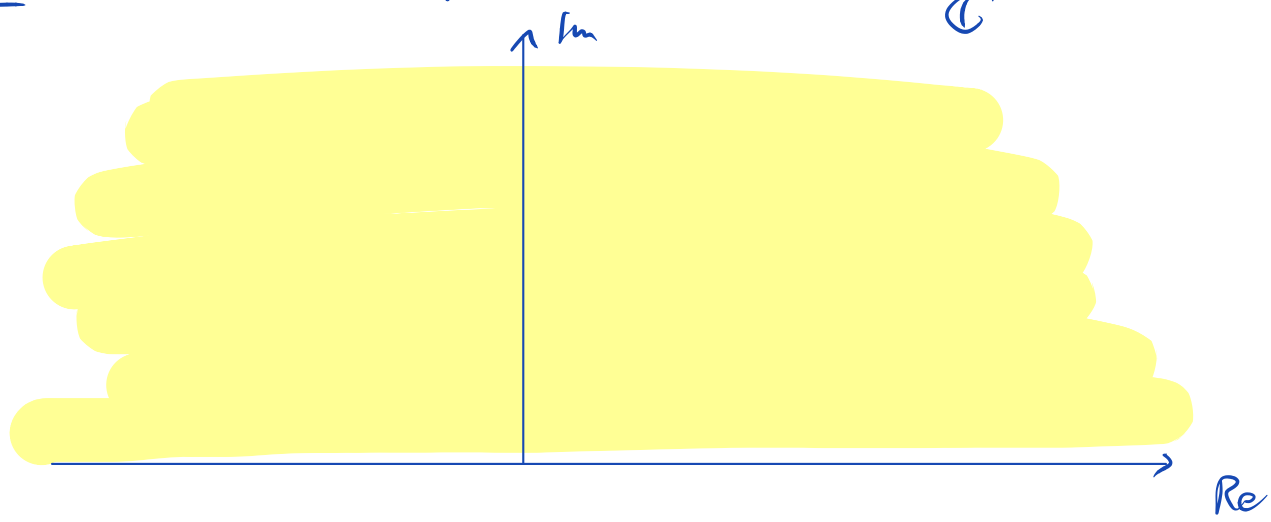
$\Theta := d_e\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$

$\mathfrak{k} = E_{-1}(\Theta) = +1$ eigenspace

$\mathfrak{p} = E_{+1}(\Theta) = -1$ eigenspace.

II) The hyperbolic plane $\mathbb{H}^2$ as a Riem. symm. space: (Ex, I.3)

Recall: $\mathbb{H}^2 = \{ z = x + i\cdot y \in \mathbb{C} \mid y > 0 \}$



The hyperbolic metric on $\mathbb{H}^2$ is given by

$$g := \frac{1}{y^2}\left(dx^2 + dy^2\right),$$

i.e. $\forall z = x + i\cdot y \in \mathbb{H}^2 \ \forall v, w \in T_z \mathbb{H}^2 \cong \mathbb{R}^2 \cong \mathbb{C}$:

$$g_z(v,w) = \frac{1}{y^2} \langle v, w \rangle_{\mathbb{R}^2} = \frac{1}{y^2}\left(v_1 \cdot w_1 + v_2 \cdot w_2\right) = \frac{1}{y^2} \operatorname{Re}\left((v_1 + i\cdot v_2)\cdot \overline{(w_1 + i\cdot w_2)}\right)$$

Claim: $G = SL_2\mathbb{R}$ acts smoothly on $\mathbb{H}^2$ via Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . z = \frac{az+b}{cz+d} \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R} \ \forall z \in \mathbb{H}^2.$$

Moreover, the induced action on the unit tangent bundle $T^1\mathbb{H}^2 \cong \mathbb{H}^2 \times S^1$ is transitive.

Pf: We omit to check that

$$(g\cdot h).z = g.(h.z) \qquad \forall g, h \in G \ \forall z \in \mathbb{H}^2.$$

(just a computation)

As to transitivity on $\mathbb{H}^2$ observe that for $z \in \mathbb{H}^2$

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} . i = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\left(\frac{\sqrt{y}\cdot i + 0}{0 \cdot i + \sqrt{y}^{-1}}\right) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} y \cdot i$$

$$= \frac{y \cdot i + x}{0 \cdot y \cdot i + 1} = x + i \cdot y = z.$$

To prove transitivity of the induced action on $T^1\mathbb{H}^2$ we are left to show that the stabilizer subgroup $K := \mathrm{stab}(i) \leq SL_2\mathbb{R}$ acts transitively on $T_i^1\mathbb{H}^2 \cong S^1$.

Note that
$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{=:k} . i = i \iff \frac{a \cdot i + b}{c \cdot i + d} = i$$
$$\iff a \cdot i + b = d \cdot i - c$$
$$\iff a = d, \ b = -c.$$

Also, $\det(k) = ad - bc = 1$, whence $1 = a^2 + c^2$, i.e.

$$k = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \theta \in [0, 2\pi].$$

$$\implies \quad \mathrm{stab}(i) = SO_2\mathbb{R}.$$

Hence: $$dk_i(e^{i\alpha}) = k'(i) \cdot e^{i\alpha}$$

$\forall g \in G \ \forall z \in \mathbb{H}^2$:

$$g'(z) = \frac{d}{dz}\left(\frac{az+b}{cz+d}\right) = \frac{a(\cancel{cz}+d) - (\cancel{az}+b)\cdot c}{(cz+d)^2}$$

$$= \frac{\overbrace{ad - bc}^{=1}}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

$$\Longrightarrow \quad dk_i(e^{i\alpha}) = k'(i)\cdot e^{i\alpha} = \frac{1}{(\sin(\theta)\cdot i + \cos(\theta))^2}\cdot e^{i\alpha}$$

$$= \frac{1}{e^{2i\theta}}\, e^{i\alpha} = e^{i(\alpha - 2\theta)}$$

Clearly, the rotation action $SO_2\mathbb{R} \curvearrowright S^1$ is transitive.

This implies that $G \curvearrowright T^1\mathbb{H}^2$ is transitive:

Let $(z,v) \in T^1\mathbb{H}^2 \cong \mathbb{H}^2 \times S^1$.

$G \curvearrowright \mathbb{H}^2$ transitive: $\exists g \in G$ s.t. $g.z = i$

$K \curvearrowright S^1$ transitive: $\exists k \in SO_2\mathbb{R}$ s.t. $dk(dg(v)) = i$

$$\Longrightarrow \quad (k\cdot g).(z,v) = (k.g.z),\ dk(dg(v)))$$
$$= (k.i\ ,\ i) = (i,i) \in T^1\mathbb{H}^2.$$

$\square$

<u>Claim:</u> $\psi: SL_2\mathbb{R} \longrightarrow Iso^+(\mathbb{H}^2)$, $g \longmapsto (z \mapsto g.z)$ is a surjective group hom. with kernel $\{\pm I\}$.

<u>Pf:</u> (0) ψ is a homom., because $SL_2\mathbb{R} \curvearrowright \mathbb{H}^2$ is a gp. action.

(i) Möbius trafo's are isometries:

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{R}$. We compute:

Recall: $g_z(v,w) = \frac{1}{y^2}\mathrm{Re}(v\cdot \bar{w})$.

$$\mathrm{Im}(g.z) = \frac{1}{2}\left(g.z - \overline{g.z}\right) = \frac{1}{2}\left(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right)$$

$$= \frac{1}{2}\left(\frac{(az+b)(c\bar{z}+d) - (a\bar{z}+b)(cz+d)}{(cz+d)\underbrace{(c\bar{z}+d)}_{=\overline{(cz+d)}}}\right)$$

$$= \frac{1}{2}\left(\frac{ac\cancel{|z|^2}+adz+bc\bar z+\cancel{bd}-ac\cancel{|z|^2}-ad\bar z-bcz-\cancel{bd}}{|cz+d|^2}\right)$$

$$= \frac{1}{2}\,\frac{\overbrace{(ad-bc)}^{=1}(z-\bar z)}{|cz+d|^2} = \frac{\operatorname{Im}(z)}{|cz+d|^2}.$$

$\Longrightarrow \quad g^*g \overset{?}{=} g$:

$$g_{g.z}(dg_z(v), dg_z(w)) = \frac{1}{|\operatorname{Im}(g.z)|^2}\cdot \operatorname{Re}\left(g'(z)\cdot v\cdot \overline{g'(z)\cdot w}\right)$$

$$= \frac{\cancel{|cz+d|^4}}{\operatorname{Im}(z)^2}\,\underbrace{\cancel{|g'(z)|^2}}_{\frac{1}{(cz+d)^2}}\operatorname{Re}(v\cdot\bar w)$$

$$= \frac{1}{\operatorname{Im}(z)^2}\cdot\operatorname{Re}(v\cdot\bar w) = g_z(v,w).$$

(ii) Möbius trafo's are or. pres.:

$SL_2\mathbb{R}$ is conn. and $\psi : SL_2\mathbb{R} \to \operatorname{Isom}(\mathbb{H}^2)$ is cont., and $id \in \psi(SL_2\mathbb{R}) \Rightarrow \psi(SL_2\mathbb{R}) \subseteq \operatorname{Iso}^0(\mathbb{H}^2) \subseteq \operatorname{Iso}^+(\mathbb{H}^2)$.

(iii) ψ is surjective.

Let $\varphi \in \operatorname{Iso}^+(\mathbb{H}^2)$. $G \curvearrowright T^1\mathbb{H}^2$ transitive,

$\exists g \in G :\quad g\cdot(i,i) = (\varphi(i), d\varphi(i))$

$\Longrightarrow \quad g^{-1}\cdot\varphi$ is an or. pres. isometry that satisfies $d(g^{-1}\cdot\varphi)_i(i) = i$

Or pres & $\mathbb{H}^2$ is 2D $\Longrightarrow$ $d(g^{-1}\circ\varphi) \equiv id|_{T_i\mathbb{H}^2}$.

Isometries are uniquely det. by their differential:

$g = \varphi$.

(iv) $\underline{\ker(\psi) = \{\pm I\}}$: Let $k \in \ker(\psi)$. Then $k \in \mathrm{stab}(i)$.

$$k = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

By the previous reasoning:

$$\begin{aligned} id = \psi(k) &\iff i = dk_i(i) = k'(i) \cdot i = e^{i(\frac{\pi}{2} - 2\theta)} \\ &\iff 2\theta \in 2\pi\mathbb{Z} \iff \theta \in \pi\mathbb{Z} \\ &\iff k \in \{\pm I\}. \end{aligned}$$

$\square$

<u>Cor</u>: $PSL_2(\mathbb{R}) = SL_2(\mathbb{R}) / \{\pm I\} \cong \mathrm{Iso}^+(\mathbb{H}^2) = \mathrm{Iso}(\mathbb{H}^2)^0$.

<u>Geodesic Symmetry at i:</u>

We need that $d(s_i)_i = -id$

For $k_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$: $k'(i) = e^{-2\theta \cdot i} \overset{!}{=} -1 = e^{\pi i}$

$$\iff \theta \in \frac{\pi}{2} + \pi \cdot \mathbb{Z}$$

The geodesic symmetry is given by

$$s_i(z) = k_{\frac{\pi}{2}} \cdot z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot z = -\frac{1}{z}.$$

Riemannian Symmetric Pair:

We set $G = \mathrm{Iso}(\mathbb{H}^2)^0 = \mathrm{Iso}^+(\mathbb{H}^2) \cong PSL_2(\mathbb{R})$,

$K = \mathrm{stab}(i) \cong PSO_2(\mathbb{R}) = SO_2(\mathbb{R})/\{\pm I\} \cong SO_2(\mathbb{R})$

$e^{i\theta} \longmapsto e^{2\theta \cdot i}$

$$\sigma(g) = [s_i]\cdot[g]\cdot[s_i] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \dots = \begin{bmatrix} -d & c \\ b & -a \end{bmatrix} \underset{\text{mod } \{\pm I\}}{=} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

Notice that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Thus: $\sigma(g) = [(g^{-1})^T]$.

We saw this in the lecture for $SL_n\mathbb{R}/SO_n\mathbb{R}$.

Cartan decomposition: $\mathfrak{sl}_2\mathbb{R} = \mathfrak{k} \oplus \mathfrak{p}$

$\Theta = d_e\sigma : \mathfrak{sl}_2\mathbb{R} \longrightarrow \mathfrak{sl}_2\mathbb{R}$

$$\frac{d}{dt}\Big|_{t=0} \sigma(\exp(t\cdot X)) = \frac{d}{dt}\Big|_{t=0} ((e^{t\cdot X})^{-1})^T = \frac{d}{dt}\Big|_{t=0} e^{-t\cdot(X^T)}$$

$$= -X^T$$

$X \in \mathfrak{k} \Leftrightarrow X = \Theta(X) = -X^T \Leftrightarrow X \in \mathfrak{so}_2\mathbb{R}$

$X \in \mathfrak{p} \Leftrightarrow -X = \Theta(X) = -X^T \Leftrightarrow X \in \mathfrak{sl}_2\mathbb{R}$ symmetric.

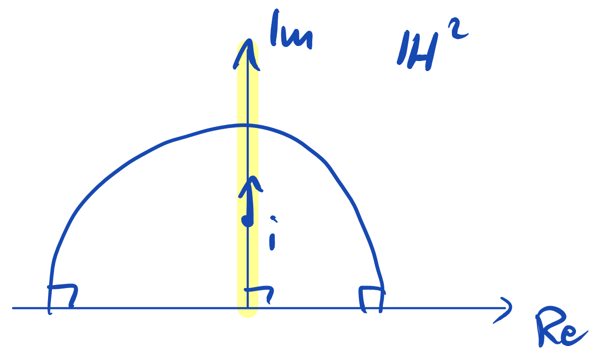
Geodesics: Thm II.21:

"Lie exp." $\quad$ Riem. exp. map

$$\begin{tikzcd} \mathfrak{p} \arrow[r, "d\pi_0"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowright"] & T_i \mathbb{H}^2 \arrow[d, "\mathrm{Exp}_i"] \\ G \arrow[r, "\pi"] & \mathbb{H}^2 \\ g \arrow[r, mapsto] & g \cdot i \end{tikzcd}$$

For example: $H := \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in \mathfrak{p}$.

$$\mathrm{Exp}_i(d\pi_0(t \cdot H)) = \pi(\exp(t \cdot H)) = \exp(t \cdot H) . i$$
$$= e^{t \cdot H} . i = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} . i$$
$$= e^t \cdot i$$



Because Möbius transformations preserve half-circles and half-lines perpendicular to the real axis, all geodesics are of this form.

III) Iwasawa decomposition of $SL_n(\mathbb{R})$:

By the QR-decomposition theorem, we can find for every $g \in G := SL_n\mathbb{R}$ a unique orthogonal matrix $k \in O_n(\mathbb{R})$ and a unique upper triangular matrix $p \in P := \left\{ \begin{pmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \,\middle|\, a_{ii} > 0 \right\}$ such that $g = k \cdot p$.

Notice: $$1 = \det(g) = \underbrace{\det(k)}_{\in\{\pm 1\}} \cdot \overbrace{\underbrace{\det(p)}_{>0}}^{= a_{11} \cdot \ldots \cdot a_{nn}} \implies \det(k) = 1$$
$$\& \; a_{11} \cdot \ldots \cdot a_{nn} = 1.$$

Moreover, $P = A \cdot N$ where $A = \left\{ \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \middle| a_{ii} > 0, \prod a_{ii} = 1 \right\}$

$N = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$.

Claim:

$\Phi : K \times P \longrightarrow G, \quad (k, p) \longmapsto k \cdot p$

$\varphi : A \times N \longrightarrow P, \quad (a, n) \longmapsto a \cdot n$

are diffeomorphisms.

Cor: (Iwasawa decomposition)

$$K \times A \times N \xrightarrow{\mathrm{id} \times \varphi} K \times P \xrightarrow{\Phi} G$$

$$(k, a, n) \longmapsto k \cdot a \cdot n$$

is a diffeomorphism.