Lecture

24 March 2021

Last week

Thm II.21 $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $M = G/K$ RSS, $G = \mathrm{Iso}(M)^\circ$, $o \in M$, $K = \mathrm{Stab}_G(o)$, $\pi : G \to G/K$. Then

$$\begin{tikzcd} \mathfrak{p} \arrow[r, "d_e\pi"] \arrow[d, "\exp"'] \arrow[dr, phantom, "\circlearrowright"] & T_oM \arrow[d, "\mathrm{Exp}_o"] \\ G \arrow[r, "\pi"'] & M \end{tikzcd}$$

$\mathrm{Exp}_o \circ d_e\pi|_{\mathfrak{p}} = \pi \circ \exp|_{\mathfrak{p}}$ or if $X \in \mathfrak{p}$
$\mathrm{Exp}_o \circ d_e\pi(X) = (\exp X)_* o$

Thm II.22 G Lie gp with $\mathrm{Lie}(G) = \mathfrak{g}$, $\exp : \mathfrak{g} \to G$. Let $X \in \mathfrak{g}$. Then if $d_X \exp : T_X\mathfrak{g} \simeq \mathfrak{g} \to T_{\exp(X)}G$ the diff, $\Rightarrow$

$$d_X \exp = d_e L_{\exp X} \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{\mathfrak{g}}(X))^n}{(n+1)!}$$

Corollary II.23 Let $\mathrm{Exp}_o \circ d_e\pi : \mathfrak{p} \to M$ and $X \in \mathfrak{p}$ $\Rightarrow$

$$d_X(\mathrm{Exp}_o \circ d_e\pi) : T_X\mathfrak{p} \simeq \mathfrak{p} \to T^M_{(\exp\circ d_e\pi)(X)}$$

is

$$d_X(\mathrm{Exp}_o \circ d_e\pi|_{\mathfrak{p}}) = (d_o L_{\exp X} \circ d_e\pi) \sum_{n=0}^{\infty} \frac{(T_X)^n}{(2n+1)!},$$

where $T_X = (\mathrm{ad}_{\mathfrak{g}}(X))^2$.

Pf $\mathrm{Exp}_o \circ d_e\pi|_{\mathfrak{p}} = \pi \circ \exp|_{\mathfrak{p}} \Rightarrow X \in$

$$d_X(\mathrm{Exp}_o \circ d_e\pi|_{\mathfrak{p}}) = d_X(\pi \circ \exp|_{\mathfrak{p}})$$

$$d_X(\pi \circ \exp) = d_{\exp(X)}\pi \circ d_X\exp \overset{\text{Thm II.22}}{=}$$

$$= \underbrace{d_{\exp(X)}\pi \circ d_e L_{\exp X}} \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{\mathfrak{g}}(X))^n}{(n+1)!}$$

$$= \underbrace{d_e(\pi \circ L_{\exp X})} \underbrace{\sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{\mathfrak{g}}(X))^n}{(n+1)!}}$$

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "L_g"'] \arrow[dr, phantom, "\circlearrowright"] & G/K \arrow[d, "L_g"] \\ G \arrow[r, "\pi"'] & G/K \end{tikzcd}$$

$$= d_e(L_{\exp X} \circ \pi) \downarrow$$

$$= d_o L_{\exp X} \circ d_e\pi \sum_{n=0}^{\infty} \frac{(-\mathrm{ad}_{\mathfrak{g}}(X))^n}{(n+1)!}$$

Recall $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{p})(\mathfrak{p}) \subset \mathfrak{k}$
($\mathfrak{p} = -1$ e.space of Θ and Θ is a Lie algebra homo)
and $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})(\mathfrak{p}) \subset \mathfrak{p}$
$\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})(\mathfrak{k}) \subset \mathfrak{k}$. Thus if $X \in \mathfrak{p}$

$$T_X(\mathfrak{p}) = \mathrm{ad}_{\mathfrak{g}}(X)\Big(\underbrace{\mathrm{ad}_{\mathfrak{g}}(X)(\mathfrak{p})}_{\subset\, \mathfrak{k}}\Big) \subset \mathfrak{p}$$

$$T_X(\mathfrak{k}) = \mathrm{ad}_{\mathfrak{g}}(X)\Big(\underbrace{\mathrm{ad}_{\mathfrak{g}}(X)(\mathfrak{k})}_{\subset\, \mathfrak{p}}\Big) \subset \mathfrak{k}$$

$\Rightarrow T_X$ preserves the Cartan decomposition $\Rightarrow$ if $Y \in \mathfrak{p}$

$$(\mathrm{ad}_{\mathfrak{g}}(X))^{2n+1}(Y) = \mathrm{ad}_{\mathfrak{g}}(X)\, T_X^{2n}(Y) \in$$

$$\subset \mathrm{ad}_{\mathfrak{g}}(X)(\mathfrak{p}) \subset \mathfrak{k} = \ker d_e\pi \Rightarrow$$

$[G \xrightarrow{\pi} G/K \qquad \mathfrak{g} \xrightarrow{d_e\pi} T_o G/K]$

$$\Rightarrow d_X(\mathrm{Exp}_o \circ d_e\pi|_{\mathfrak{p}}) =$$

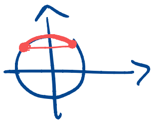
$$= d_o L_{\exp X} \circ d_e\pi \sum_{n=0}^{\infty} \frac{(T_X)^n}{(2n+1)!} \qquad \boxtimes$$

II.6 Totally geodesic submanifolds.

Defn Let $N \subset M$ be a submanifold of a Riem. mfd (M, g). We say that N is geodesic at $p \in N$ if $\forall\, v \in T_pN$ the M-geodesic through p with tangent vector v is all contained in N.
We say that N is totally geodesic if it is geodesic at every $p \in N$.

(M, g) Riem. mfd, $N \subset M$ subm. $\Rightarrow g|_N$ is a Riem. metric on N

A priori, if $p,q\in N \Rightarrow$
$\Rightarrow d_M(p,q)\le d_N(p,q)$,
where d_M, d_N are the distances induced by g & $g|_N$.



Fact Assume $N\subset M$ t.g.

(1) The inclusion $(N,d_N)\hookrightarrow(M,d_M)$ is locally distance preserving

(2) Every N-geodesic is an M-geodesic and every M-geodesic contained in N is an N-geodesic.

Examples

(1) In $\mathbb{R}^n$ all linear subspaces and their translates are totally geodesic. ($S^2\subset\mathbb{R}^2$ not tot. geod.)

(2) In S^n the tot. geod. subspaces are the intersection of S^n with a linear subspace of $\mathbb{R}^{n+1}$.

(3) (Cartan) M Riem. mfd s.t. $\forall p\in M$ & $\forall$ 2-dim. plane $P\subset T_pM$ $\exists$ a tot. geod. subm. through p tg to P $\Rightarrow$ M constant curvature.

Thm II.24 Let (M,g) be a Riem. mfd and $N\subset M$ a connected submfd. Then N is totally geodesic iff the parallel transport w.r.t. g along curves in N preserves the tangent spaces (i.e. parallel transport preserves $\{T_pN : p\in N\}$)

Being totally geod. is local

Example $\mathbb{T}^n=\mathbb{E}^n/\mathbb{Z}^n$, $\pi:\mathbb{E}^n\to\mathbb{T}^n$.
If $P\subset\mathbb{E}^n$ is k-dim. subspace, $k<n$
$\Rightarrow \pi(P)$ is a tot. geod. sbmf of $\mathbb{T}^n$.
P can be chosen in such a way that $\pi(P)$ is dense in $\mathbb{T}^n$. (e.g. $n=2$, P irrational line)

Defn. A subspace $\mathfrak{m}$ of a Lie algebra $\mathfrak{g}$ is a Lie triple system if $\forall X,Y,Z\in\mathfrak{m}$, $[[X,Y],Z]\in\mathfrak{m}$.

Ex $\mathfrak{p}\subset\mathfrak{g}$, since $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$ and $[\mathfrak{p},\mathfrak{k}]\subset\mathfrak{p}$.

Theorem II.25 $M=G/K$ RSS, $o\in M$ basept, $K=\mathrm{Stab}_G(o)$ where $G=\mathrm{Iso}(M)^\circ$.
Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan dec.

(1) If $\mathfrak{m}\subset\mathfrak{p}$ is a Lie triple system, then $N:=(\mathrm{Exp}_o\circ d_e\pi)(\mathfrak{m})\subset M$ is a totally geodesic submanifold through $o\in M$ and such that $T_oN=d\pi(\mathfrak{m})$.

(2) If $N\subset M$ is a t.g. sbmf. through o, then $\mathfrak{m}:=(d_e\pi)^{-1}(T_oN)$ is a Lie triple system.

Rk If $N\subset M$ is a t.g. sbmf. Let $p\in N$ and $g\in G$ s.t. $go=p$.
Then $L_g^{-1}(N)$ is a tot. geod. sbmf. through o to which one can apply the thm.

Lemma If $\mathfrak{m}\subset\mathfrak{g}$ is L.t.s. $\Rightarrow$

(1) $[\mathfrak{m},\mathfrak{m}]$ is a subalgebra

(2) $\mathfrak{m}+[\mathfrak{m},\mathfrak{m}]$ is a subalgebra.

Pf (1) Want that if $X,Y,Z,W\in\mathfrak{m}$
$\Rightarrow [[X,Y],[Z,W]]\in[\mathfrak{m},\mathfrak{m}]$

$$[[X,Y],[Z,W]] \underset{\uparrow}{=}$$

Jacobi identity to $[X,Y]$, Z & W

$$= -[Z,[W,[X,Y]]]-[W,[[X,Y],Z]]$$

(Jacobi identity: $\mathrm{ad}(x)$ is a deriv.)

$[X,Y]\in[\mathfrak{m},\mathfrak{m}]\subset\mathfrak{m}$ $\Rightarrow$ $[\mathfrak{m},\mathfrak{m}]\not\subset\mathfrak{m}$

$\Rightarrow [W,[X,Y]]\cancel{\subset[\mathfrak{m},\mathfrak{m}]}\subset\mathfrak{m}$

$\Rightarrow [Z,\underbrace{[W,[X,Y]]}_{\in\,\mathfrak{m}}]\subset[\mathfrak{m},\mathfrak{m}]$

Likewise for $[W, [[X,Y],Z]]$.

(2) $[\mathfrak{n} + [\mathfrak{n},\mathfrak{n}], \mathfrak{n} + [\mathfrak{n},\mathfrak{n}]] \subset$

$\subset [\mathfrak{n},\mathfrak{n}] + \overbrace{[\mathfrak{n}, [\mathfrak{n},\mathfrak{n}]]}^{\subset \mathfrak{n}} +$

$+ [[\mathfrak{n},\mathfrak{n}], [\mathfrak{n},\mathfrak{n}]] \quad \cancel{\mathfrak{n}} \quad \subset$

$([\mathfrak{n},\mathfrak{n}] \subset \mathfrak{k})?$

$\subset [\mathfrak{n},\mathfrak{n}] \,\&\, \mathfrak{n} + [\mathfrak{n},\mathfrak{n}]$.

Pf of Thm II.25

(1) $\mathfrak{n} + [\mathfrak{n},\mathfrak{n}] \subset \mathfrak{g}$ is a subalgebra and $\mathfrak{g} = \mathrm{Lie}(G)$.
Let G' be the connected Lie sbgp s.t. $\mathrm{Lie}(G') = \mathfrak{n} + [\mathfrak{n},\mathfrak{n}] =: \mathfrak{g}'$.
Let $G' \longrightarrow M$ and let
$g \longmapsto g_* o$

$K' := K \cap G'$. The inclusion $G' \hookrightarrow G$ is cont. $\Rightarrow K'$ is closed in G'.

$\Rightarrow$ We can give $M' := G' \cdot o$ the topology & diff. structure of G'/K'.

$\Rightarrow M' \subset M$ is a submanifold and $o \in M' \subset M$ is a basept.

We claim that $T_o M' = d_e\pi(\mathfrak{n})$.

Since $M' = G'/K'$ and

$\mathfrak{g}' = \mathfrak{n} + [\mathfrak{n},\mathfrak{n}] = \mathrm{Lie}(G')$

all we need to show is that $\mathrm{Lie}(K') = [\mathfrak{n},\mathfrak{n}]$.

In fact $K' = K \cap G' \Rightarrow$

$\Rightarrow \mathrm{Lie}(K') = \mathfrak{k} \cap (\mathfrak{n} + [\mathfrak{n},\mathfrak{n}]) =$

$= [\mathfrak{n},\mathfrak{n}]$

since • $\mathfrak{n} \subset \mathfrak{p}$ hence $\mathfrak{k} \cap \mathfrak{n} = (0)$

• $[\mathfrak{n},\mathfrak{n}] \subset \mathfrak{k}$.

So, given a Lie triple system $\mathfrak{n}$, we found a sbmfd $M' \subset M$ whose tg space is the Lie triple system. We need to show that M' is totally geodesic.

Let $X \in \mathfrak{n}$, $v = (d_e\pi)(X) \in T_o M'$.

The M-geod. through o with tg. vector v is $t \longmapsto \exp(tX)_*(o) =$

$= \mathrm{Exp}_o(tv)$.

But $\forall t \in \mathbb{R}$ $tX \in \mathfrak{n} \Rightarrow$

$\Rightarrow \exp(tX) \in G' \Rightarrow \exp(tX)_* \cdot o \in M'$

Hence M' is totally geodesic.

(2) To show: If $N \subset M$ is tot. geod $\overset{?}{\Rightarrow}$ $\mathfrak{n} := (d_e\pi)^{-1}(T_o N)$ is a Lie triple system.

Claim If $X, Y \in \mathfrak{n}$ then $T_X(Y) \in \mathfrak{n}$.

$([X,[X,Y]] \in \mathfrak{n})$

Want to show that if $X, Y, Z \in \mathfrak{n}$ $\Rightarrow [[X,Y],Z] \in \mathfrak{n}$.

In particular

$\mathfrak{n} \ni T_{Y+Z}(X) = \mathrm{ad}_{\mathfrak{g}}(Y+Z)\left(\mathrm{ad}_{\mathfrak{g}}(Y+Z)(X)\right)$

$= [Y+Z, [Y+Z, X]] =$

$= [Y+Z, [Y,X] + [Z,X]] =$

$= [Y, [Y,X]] + [Y, [Z,X]] +$

$+ [Z, [Y,X]] + [Z, [Z,X]] =$

$= T_Y(X) + [Y, [Z,X]] +$

$+ [Z, [Y,X]] + T_Z(X)$

$\Rightarrow [Y, [Z,X]] + [Z, [Y,X]] =$

$= T_{Y+Z}(X) - T_Y(X) - T_Z(X) \in \mathfrak{n}$

But

$$① \; [Y,[Z,X]] + \underbrace{[Z,[Y,X]]}_{\substack{\| \\ -[Y,[X,Z]] - [X,[Z,Y]] \\ = [Y,[Z,X]] + [X,[Y,Z]]}} =$$

$$= [Y,[Z,X]] + [Y,[Z,X]] + [X,[Y,Z]] =$$

$$= 2[Y,[Z,X]] + [X,[Y,Z]] \in \mathfrak{n}$$

Exchange X & Y

$$② \; [X,[Z,Y]] + [Z,[X,Y]] =$$

$$= 2[X,[Z,Y]] + [Y,[X,Z]] \in \mathfrak{n}$$

$$\mathfrak{n} \ni ① - ② =$$

$$= 2[Y,[Z,X]] + [X,[Y,Z]] - \left(2[X,[Z,Y]] + [Y,[X,Z]]\right) =$$

$$= 3[Y,[Z,X]] + [X,[Y,Z]] - +2[X,[Z,Y]] =$$

$$= 3[Y,[Z,X]] + 3[X,[Y,Z]] =$$

$$= 3[X,[Y,Z]] \quad \boxed{\text{W}}$$

assuming the claim.