lecture

25 March 2021

Thm II.25 $M = G/K$ Rss, $o \in M$, $K = \mathrm{Stab}_G(o)$, $G = \mathrm{Iso}(M)^\circ$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition.

(1) If $\mathfrak{m} \subset \mathfrak{p}$ Lie triple system $\Rightarrow$ $N := \exp(\mathfrak{m})$ is a t.g. subm. and $T_oN = \mathfrak{m}$.

(2) $N \subset M$ t.g. sbmf. through o $\Rightarrow$
$\Rightarrow$ $\mathfrak{m} := d_e\pi(T_oN)$ is a Lie triple system.

Yesterday: **Claim** $\forall X, Y \in \mathfrak{m} \Rightarrow$
$\Rightarrow T_X Y = (\mathrm{ad}_\mathfrak{g}(X))^2(Y) \in \mathfrak{m}$.

With this we completed the proof.

Pf of Claim $\mathfrak{m} := d_e\pi(T_oN)$.

$X \in \mathfrak{m} \Rightarrow d_e\pi(X) \in T_oN \Rightarrow$
$t \longmapsto (\exp tX)_* o = (\mathrm{Exp}_o \circ d_e\pi\, X)$
is an M-geodesic through o s.t. the tg vector at o is

$d_e\pi(X) \in T_oN$. N tot.g. $\Rightarrow$
$t \longmapsto \mathrm{Exp}_o \cdot d_e\pi(X) \subset N \quad \forall t \in \mathbb{R}$

$$\mathfrak{m} \xrightarrow{d_e\pi} T_oN \xrightarrow{\mathrm{Exp}_o} N$$

$$d_{tX}(\mathrm{Exp}_o \circ d_e\pi): T_{tX}\mathfrak{m} = \mathfrak{m} \longrightarrow T_{(\mathrm{Exp}_o \circ d_e\pi)(tX)} N$$

Corollary II.23 $\Rightarrow$ $\forall Y \in \mathfrak{m}$

$$d_{tX}(\mathrm{Exp}_o \circ d_e\pi)(Y) = d_o L_{\exp(tX)} \cdot d_e\pi \left(\sum_{n=0}^{\infty} \frac{(T_{tX})^n(Y)}{(2n+1)!} \right)$$

$$\Rightarrow (d_o L_{\exp(tX)})^{-1} \underbrace{d_{tX}(\mathrm{Exp}_o \circ d_e\pi)(Y)}_{\substack{\cap \\ T_{(\mathrm{Exp}_o \circ d_e\pi)(tX)}}} = d_e\pi \sum_{n=0}^{\infty} \frac{(T_{tX})^n(Y)}{(2n+1)!}$$

Want to see that $d_e\pi \sum_{n=0}^{\infty} \frac{(T_{tX})^n(Y)}{(2n+1)!} \in T_oN$.

Recall:

- (Proof of Thm II.21 - rel. between $\exp$ & Exp) $t \longmapsto \exp(tX)$ is a geod & let τ_t be the translation along it. τ_t is a 1-par. sbgp.
$\Rightarrow \tau_t = \exp(tX)$.

- Proposition II.14 If τ_t is the transvection along γ
($\tau_t = s_{\gamma(t/2)} \cdot s_{\gamma(0)}$) $\Rightarrow$
$\Rightarrow d\tau_t : T_{\gamma(0)} \longrightarrow T_{\gamma(t)}$
is the parallel transport along γ.

$\Rightarrow (d_o L_{\exp tX})^{-1}$ is the parallel transport along $t \longmapsto \exp tX \in N$
& N is tot. geod $\Rightarrow$ the geod. is all contained in N.

Thm II.24 $\Rightarrow$ parallel transport preserves $\{T_pN : p \in N\}$ $\Rightarrow$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(T_{tX})^n(Y)}{(2n+1)!} \in \mathfrak{m},$$

We write

$$\varphi(t) := \sum_{n=0}^{\infty} \frac{(T_{tX})^n(Y)}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{((\mathrm{ad}_\mathfrak{g}(tX))^2)^n(Y)}{(2n+1)!} =$$

$$= t^2 \frac{\mathrm{ad}_\mathfrak{g}(X)^2(Y)}{3!} + t^4(\cdots)$$

$\mathfrak{m}$
$\downarrow$

$$\varphi''(t)\Big|_{t=0} = \frac{1}{3}(\mathrm{ad}_\mathfrak{g}(X))^2(Y) = \frac{1}{3} T_X(Y) \qquad \blacksquare$$

Remark $\mathfrak{m} \subset \mathfrak{g}$ Lie triple system with t.g. sbmf. $N = \mathrm{Exp}(\mathfrak{m})$ through o. Let $\mathfrak{g}' = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ is a subalgebra of $\mathfrak{g}$ with Lie subgroup $G' < G$. Set $K' := K \cap G'$. Let $\Theta = d_e\sigma$ be the Cartan involution $\Rightarrow$
$\Theta X = -X \quad \forall X \in \mathfrak{m} \subset \mathfrak{p}$.

Moreover

$$\Theta([\mathfrak{m}, \mathfrak{m}]) = [\Theta(\mathfrak{m}), \Theta(\mathfrak{m})] \subset [\mathfrak{m}, \mathfrak{m}] \Rightarrow \Theta(\mathfrak{g}') = \mathfrak{g}'$$

and also $\sigma(G') \subset G'$.

Let $\sigma' := \sigma|_{G'}$ be an involution of G'. Want to show that $((G')^{\sigma'})^\circ \subseteq K' \subseteq (G')^{\sigma'}$.

If so then (G', K') is a RSP associated to N $\Rightarrow$

$\Rightarrow$ N is a RSS, $N \cong G'/K'$

$$K' = K \cap G' \subseteq (G^\sigma) \cap G' = (G')^{\sigma'}$$

$$K' = K \cap G' \supseteq (G^\sigma)^\circ \cap G'$$

But $(G^\sigma)^\circ \cap G'$ is an open subgp of G' $\Rightarrow$

$$(G^\sigma)^\circ \cap G' \supseteq ((G')^{\sigma'})^\circ \quad ✓$$

II.7 Example

Riemann sym. space for the sym. pair $(SL(n,\mathbb{R}), SO(n,\mathbb{R}))$

$\sigma \in \mathrm{Aut}(G)$, $G := SL(n,\mathbb{R})$, $K = SO(n,\mathbb{R})$, $\sigma(g) = {}^t g^{-1}$ is the involution with $K = G^\sigma$. $SO(n,\mathbb{R})$ cpt $\Rightarrow$

$\Rightarrow$ $(SL(n,\mathbb{R}), SO(n,\mathbb{R}))$ RSP.

$SL(n,\mathbb{R}) < GL(n,\mathbb{R})$ $\Rightarrow$

$\Rightarrow$ $\exp : \mathfrak{sl}(n,\mathbb{R}) \to SL(n,\mathbb{R})$ is the matrix exp.

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!} \Rightarrow$$

$\Rightarrow$ $\exp({}^t X) = {}^t(\exp(X))$

$\Rightarrow$ $\sigma(\exp tX) = \exp(-t\,{}^t X)$

$\Rightarrow$ $\Theta = d_e\sigma$ is

$$\Theta(X) = \frac{d}{dt}\Big|_{t=0} \exp(-t\,{}^t X) = -{}^t X$$

$$\mathfrak{sl}(n,\mathbb{R}) = \{X \in \mathfrak{gl}(n,\mathbb{R}) : \mathrm{tr}\, X = 0\}$$

$$\mathfrak{k} = \{X \in \mathfrak{sl}(n,\mathbb{R}) : \Theta(X) = X\}$$
$$= \{X \in \mathfrak{sl}(n,\mathbb{R}) : -{}^t X = X\}$$

$$\mathfrak{p} = \{X \in \mathfrak{sl}(n,\mathbb{R}) : {}^t X = X\}$$

$\Rightarrow$ $\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{k} \oplus \mathfrak{p}$

$X \in \mathfrak{sl}(n,\mathbb{R})$ $\Rightarrow$

$$\Rightarrow X = \underbrace{\tfrac{1}{2}(X - {}^t X)}_{\in\, \mathfrak{k}} + \underbrace{\tfrac{1}{2}(X + {}^t X)}_{\in\, \mathfrak{p}}$$

Want an $\mathrm{Ad}_G(K)$-inv. inner product on $\mathfrak{p}$.

Recall that

$$\mathrm{Ad}_G : G \to GL(\mathfrak{g})$$

is conjugation since $G < GL(n,\mathbb{R})$,

$$\mathrm{Ad}_G(g)(X) = gXg^{-1}$$

To get such inner product

$$M_{n\times n} \times M_{n\times n} \to \mathbb{R}$$
$$(A, B) \mapsto \mathrm{tr}({}^t A B)$$

(If $M_{n\times n} \cong \mathbb{R}^{n^2}$, this is the usual scalar product on $\mathbb{R}^{n^2}$). This is clearly $\mathrm{Ad}_G(O(n,\mathbb{R}))$-inv.

However on $\mathfrak{p}$ the inner product is

$$(A,B) \longmapsto \mathrm{tr}(AB)$$

Consider the model $\mathcal{P}'(n)$ for $SL(n,\mathbb{R})/SO(n,\mathbb{R})$

$$\mathcal{P}'(n) := \{ S \in M_{n\times n}(\mathbb{R}) : {}^tS = S, \ \det S = 1, \ S \gg 0 \}$$

where $SL(n,\mathbb{R}) \curvearrowright \mathcal{P}'(n)$ via $g_* S := g S {}^t g$.

<u>Notation</u> Take $I \in \mathcal{P}'(n)$ as basepoint. We noted that $(\mathrm{Exp}_I \circ d_e\pi)(X) = (\exp X)_* I$

$$\mathrm{Exp} := \mathrm{Exp}_I \circ d_e\pi$$

$$\begin{tikzcd} \mathfrak{p} \arrow[r, "d_e\pi"] \arrow[rr, bend right, "\mathrm{Exp}"'] & T_I \mathcal{P}'(n) \arrow[r, "\mathrm{Exp}_I"] & \mathcal{P}'(n) \end{tikzcd}$$

<u>Fact</u>: $\mathrm{Exp} : \mathfrak{p} \to \mathcal{P}'(n)$ diffeom. and if we consider $\mathrm{Exp}(0), \mathrm{Exp}(X) \in \mathcal{P}'(n)$ $\exists$ a unique geodesic between these two pts, $t \longmapsto \mathrm{Exp}(tX)$ and this geod. is length minimizing $\Rightarrow$

$$\Rightarrow d(\mathrm{Exp}(X), \mathrm{Exp}(0)) = \|X\|$$

Let $X \in \mathfrak{p}$. Then

$(\cdot)$ $\mathrm{Exp}(X) = (\exp X)_* I = \exp(X)\, I \exp({}^tX) = \exp(2X)$.

$(\because)$ $\exp(-X)_* \mathrm{Exp}(X) = \exp(-X)\mathrm{Exp}(X)\exp(-X) = \exp(-X)\exp(2X)\exp(-X) = I \in \mathcal{P}'(n)$

Let $\mathfrak{a} := \{ \mathrm{diag}(x_1, \dots, x_n) : \sum x_i = 0 \}$

Then $\mathfrak{a} \subset \mathfrak{p}$ since $({}^t\mathfrak{a} = \mathfrak{a})$ and

$[\mathfrak{a}, \mathfrak{a}] = 0 \Rightarrow \mathfrak{a}$ is a Lie triple system $\Rightarrow$

$$\Rightarrow F := \mathrm{Exp}(\mathfrak{a}) = \{ \mathrm{diag}(x_1, \dots, x_n) : \prod x_i = 1 \}$$

is a totally geodesic submfd.

Want to compute the distance in F. $X_1, X_2 \in \mathfrak{a}$, $\mathrm{Exp}\, X_i \in F$

$$d(\mathrm{Exp}(X_1), \mathrm{Exp}(X_2)) \overset{(\because)\ \&\ SL(n,\mathbb{R})\text{-inv. of } d}{=} d(\exp(-X_2)_* \mathrm{Exp}(X_1), I)$$

$$= d(\exp(-X_2)\exp(2X_1)\exp(-X_2), I)$$

$$\underset{\substack{X_1, X_2 \\ \text{commute}}}{=} d(\exp(2X_1 - 2X_2), I) \overset{\text{Fact}}{=} \|X_1 - X_2\|$$

$\Rightarrow$ $\mathrm{Exp} : \mathfrak{a} \to F$ is an isometry. $\mathfrak{a} \subset \mathbb{R}^n$

$\Rightarrow$ we call F a flat and $\dim F = n-1$.

$\mathfrak{sl}(n,\mathbb{R}) \supset$ max. abelian subalgebra that is diagonalizable ($= \mathfrak{a}$)

$\dim \mathfrak{a} = \mathrm{rk}\, \mathfrak{sl}(n,\mathbb{R})$

This is the same as the max. dim. of a flat.

rk	1	2	3	...

$\underbrace{2 \quad 3 \quad \ldots}_{\text{higher rank}}$

$[\mathfrak{a},\mathfrak{a}] = 0 \Rightarrow \mathfrak{a}$ is a Lie triple system $\Rightarrow$

$\Rightarrow F := \mathrm{Exp}(\mathfrak{a}) =$

$$= \{\mathrm{diag}(x_1, \ldots, x_n) : \textstyle\prod x_i = 1\}$$

is a totally geodesic submfd.

Want to compute the distance in F. $X_1, X_2 \in \mathfrak{a}$, $\mathrm{Exp}\, X_i \in F$

$$d(\mathrm{Exp}(X_1), \mathrm{Exp}(X_2)) \overset{(*)\ \&\ SL(n,\mathbb{R})\text{-inv. of } d}{=}$$

$$= d(\exp(-X_2)_* \mathrm{Exp}(X_1), I)$$

$$= d(\exp(-X_2)\exp(2X_1)\exp(-X_2), I)$$

$$\underset{\substack{X_1, X_2 \\ \text{commute}}}{=} d(\exp(2X_1 - 2X_2), I)$$

$$\overset{\text{Fact}}{=} \|X_1 - X_2\|$$