lecture

31 March 2021

I.8 Decomposition of symm. spaces

To show: If M is a RSS $\Rightarrow$ M is the product of (possibly) three kinds of RSS.

- Euclidean spaces
- RSS of cpt type
- RSS of non-cpt type

RSS $\rightsquigarrow$ RSP $\rightsquigarrow$ RSS
$\Updownarrow$
orthogonal symmetric Lie algebras

M RSS, $o \in M$, $M = G/K$, $G = \mathrm{Iso}(M)^0$ is a Lie gp $\rightsquigarrow$ $\mathfrak{g} = \mathrm{Lie}(G)$.
~~$G \xrightarrow{\pi} G/K \rightsquigarrow d_e\pi : \mathfrak{g} \to T_oM$~~

RSP (G, σ), $d_e\sigma =: \Theta$

$(\mathfrak{g}, \Theta)$ the prototype of OSLA

Recall The Killing form of a Lie algebra $\mathfrak{g}$ is a bilinear symm. form

$$B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow k = \text{field of defn. of } \mathfrak{g}$$

$$B_{\mathfrak{g}}(X,Y) := \mathrm{tr}(\mathrm{ad}_{\mathfrak{g}}(X)\,\mathrm{ad}_{\mathfrak{g}}(Y))$$

Properties (1)

If $\alpha \in \mathrm{Aut}(\mathfrak{g})$ $\Rightarrow$

$\Rightarrow$ $B_{\mathfrak{g}}(\alpha(X), \alpha(Y)) = B_{\mathfrak{g}}(X,Y)$ $\forall\, X, Y \in \mathfrak{g}$.

(2) If $D \in \mathrm{Der}(\mathfrak{g})$,
(that is $D[X,Y] = [DX,Y] + [X,DY]$)

$\Rightarrow$ $B_{\mathfrak{g}}(DX, Y) + B_{\mathfrak{g}}(X, DY) = 0$

In particular if $Z \in \mathfrak{g}$ $\Rightarrow$ $\mathrm{ad}_{\mathfrak{g}}(Z) \in \mathrm{Der}(\mathfrak{g})$ and hence

$$B_{\mathfrak{g}}(\mathrm{ad}_{\mathfrak{g}}(Z)X, Y) + B_{\mathfrak{g}}(X, \mathrm{ad}_{\mathfrak{g}}(Z)Y) = 0$$

Back to the properties of $(\mathfrak{g}, \Theta)$: related to a RSP (G,K).

(1) Θ is an involution of $\mathfrak{g}$ and $\mathrm{Lie}(K) = \mathfrak{k}$ is the eigenspace with e.value 1.

(2) $\mathrm{ad}_{\mathfrak{g}} = d_e \mathrm{Ad}_G$. Since K cpt.

$\Rightarrow$ $\mathrm{Ad}_G(K) < GL(\mathfrak{g})$ cpt. sbgp.

$$\mathrm{Lie}(\mathrm{Ad}_G(K)) = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{k}).$$

Defn. $\mathfrak{g}$ Lie algebra. We say that a subalgebra $\mathfrak{u} \subset \mathfrak{g}$ is **compactly embedded** if $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{u}) \subset \mathfrak{gl}(\mathfrak{g})$ is the Lie algebra of a cpt subgroup $U < GL(\mathfrak{g})$.

We would like to say that $U = \mathrm{Ad}_G(K)$, $K \subset G$ cpt. and $K \cong U$ but K can't have a center

$$U = \mathrm{Ad}_G(K) \big/ Z(G) \cap K$$

Fact Any such $U < \mathrm{Aut}(\mathfrak{g})$

Pf By naturality $\forall t \in \mathbb{R}, X \in \mathfrak{g}$

$$\mathrm{Ad}_G(\exp tX) = \exp(\mathrm{ad}_{\mathfrak{g}}(tX))$$

But $\mathrm{Ad}_G = d_e c_g$ $\Rightarrow$

$\Rightarrow$ $\mathrm{Ad}_G(g) \in \mathrm{Aut}(\mathfrak{g})$ $\forall g \in G$

$\Rightarrow$ $\underbrace{\mathrm{Lie}(U) = \mathrm{ad}_{\mathfrak{g}}(\mathfrak{u}) \subset \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))}$
~~only needed this~~, not cpty emb.

$\Rightarrow$ $U^0 < \mathrm{Aut}(\mathfrak{g})$, U conn $\Rightarrow$

$\Rightarrow$ $U \subset \mathrm{Aut}(\mathfrak{g})$. ∎

Definition (i) An orthogonal symmetric Lie algebra (OSLA) is a pair $(\mathfrak{g}, \Theta)$ such that
(1) $\mathfrak{g}$ is a real Lie algebra;
(2) Θ is an involution of $\mathfrak{g}$ such that the subalgebra $\mathfrak{u} := \{X \in \mathfrak{g} : \Theta X = X\}$ of fixed pts of Θ is compactly embedded.

Θ involution $\Rightarrow$ $\Theta^2 = Id$ $\Rightarrow$
$\Rightarrow$ e.v. ± 1

$$\mathfrak{u} = \{X \in \mathfrak{g} : \Theta X = X\}$$

$$\mathfrak{e} = \{X \in \mathfrak{g} : \Theta X = -X\}$$

Lemma II.26 (1) The decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ is orthogonal w.r.t. $B_{\mathfrak{g}}$.
(2) If $\mathfrak{g}$ is effective $\Rightarrow$
$\Rightarrow B_{\mathfrak{g}}|_{\mathfrak{u}\times\mathfrak{u}}$ is negative definite

Part (2) of the defn.
$\mathfrak{g}$ is effective if $\mathfrak{u} \cap \mathfrak{z}(\mathfrak{g}) = (0)$

Pf of lemma (1) $X \in \mathfrak{u}, Y \in \mathfrak{e}$, so that $\Theta X = X$ $\Theta Y = -Y$.

$$\Rightarrow B_{\mathfrak{g}}(X, Y) = B_{\mathfrak{g}}(\Theta X, \Theta Y) =$$

$$= - B_{\mathfrak{g}}(X, Y) \Rightarrow B_{\mathfrak{g}}(X, Y) = 0.$$

(2) $(\mathfrak{g}, \Theta)$ OSLA, $ad_{\mathfrak{g}}(\mathfrak{u}) = Lie(U)$
$U < GL(\mathfrak{g})$ cpt Lie gp. Let $\langle , \rangle$ be an inner product on $\mathfrak{g}$ that is U-invariant $\Rightarrow$
$\Rightarrow$ $U \subset O(\mathfrak{g}, \langle , \rangle)$

$\Rightarrow Lie(U) = ad_{\mathfrak{g}}(\mathfrak{u}) \subset \mathfrak{o}(\mathfrak{g}, \langle , \rangle)$
that is elements in $ad_{\mathfrak{g}}(\mathfrak{u})$ are skew-sym. w.r.t. $\langle , \rangle$ $\Rightarrow$
if $X \in \mathfrak{u}$, $\{e_1, \dots, e_n\}$ basis of $\mathfrak{g}$ $\Rightarrow$

$$\Rightarrow B_{\mathfrak{g}}(X, X) = tr(ad_{\mathfrak{g}}(X)^2) =$$

$$= \sum_{j=1}^{n} \langle ad_{\mathfrak{g}}(X)^2 e_j, e_j \rangle =$$

$$= - \sum_{j=1}^{n} \langle ad_{\mathfrak{g}}(X) e_j, ad_{\mathfrak{g}}(X) e_j \rangle$$

$$= - \sum_{j=1}^{n} \| ad_{\mathfrak{g}}(X) e_j \|^2 \leq 0$$

with equality $\Leftrightarrow$ $ad_{\mathfrak{g}}(X) = 0$ that $X \in \mathfrak{u} \cap \mathfrak{z}(\mathfrak{g}) = (0)$ ∎

Defn. Let $(\mathfrak{g}, \Theta)$ be an effective OSLA with Killing form $B_{\mathfrak{g}}$ and decomp $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$
(1) $(\mathfrak{g}, \Theta)$ is of compact type if $B_{\mathfrak{g}}$ is negative defn.
(2) $(\mathfrak{g}, \Theta)$ is of non-compact type if $B_{\mathfrak{g}}|_{\mathfrak{e}}$ is positive-defn.
(3) $(\mathfrak{g}, \Theta)$ is of Euclidean type if $\mathfrak{e}$ is an Abelian ideal.

Recall / Remark

- $\mathfrak{g}$ Lie algebra is simple if
  - $\mathfrak{g}$ not Abelian
  - $\nexists$ no non-trivial ideals
- $\mathfrak{g}$ Lie alg. is semisimple if it is direct sum of simple ideals. $\mathfrak{g} = \sum^{\oplus} \mathfrak{g}_j$

- $\mathfrak{g}$ is semisimple $\Leftarrow$ $B_\mathfrak{g}$ is non-degenerate.

(i) In cases (1) & (2) $\Rightarrow$ $\mathfrak{g}$ is semisimple

(ii) $(\mathfrak{g}, \theta)$ Euclidean iff $[\mathfrak{e}, \mathfrak{e}] = 0$ ($\Leftarrow$ easy to check that $[\mathfrak{g}, \mathfrak{e}] \subset \mathfrak{e}$).

Thm II.27 (Decomposition thm for OSLA) $(\mathfrak{g}, \theta)$ effective OSLA. Then $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ is a decomp. into θ-invariant ideals s.t. $(\mathfrak{g}_-, \theta|_{\mathfrak{g}_-})$ is of non-cpt. type, $(\mathfrak{g}_0, \theta|_{\mathfrak{g}_0})$ is Euclidean & $(\mathfrak{g}_+, \theta|_{\mathfrak{g}_+})$ is of cpt. type. Moreover the decomp. is orth. w.r.t. $B_\mathfrak{g}$.

How to construct $\mathfrak{g}_+, \mathfrak{g}_0, \mathfrak{g}_-$

$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ is a U-invariant decomposition $\Rightarrow$ let $\langle , \rangle$ be a U-invariant inner product on $\mathfrak{e}$. Since $B_\mathfrak{g}|_\mathfrak{e}$ is a sym. bilinear form $\Rightarrow \exists!$ $A \in End(\mathfrak{e})$ symmetric s.t.

$$B_\mathfrak{g}(X, Y) = \langle AX, Y \rangle \quad \forall X, Y \in \mathfrak{e}$$

we said already that $U \subset Aut(\mathfrak{g})$. Since $B_\mathfrak{g}$ is U-invariant $\Rightarrow \forall X, Y \in \mathfrak{e}$, $k \in U$

$$\Rightarrow B_\mathfrak{g}(kX, kY) = B(X, Y)$$

$$\Rightarrow \langle AkX, kY \rangle = \langle AX, Y \rangle$$

$$\overset{\text{U-inv. } \langle , \rangle}{\Rightarrow} \langle k^{-1}AkX, Y \rangle = \langle AX, Y \rangle$$

$$\Rightarrow Ak \overset{(*)}{=} kA \Rightarrow$$

$$\Rightarrow A \cdot ad_\mathfrak{g}(X) \overset{(*)}{=} ad_\mathfrak{g}(X) \cdot A \quad \forall X \in \mathfrak{u}$$

A sym. $\Rightarrow$ o.n. basis of $\mathfrak{e}$ $\{f_1, \dots, f_n\}$ consisting of e.v. of A with e.v. $\{\beta_1, \dots, \beta_n\}$.

Since (*) they are also preserved by U and $ad_\mathfrak{g}(\mathfrak{u})$.

Define the following subspaces of $\mathfrak{e}$.

$$\mathfrak{e}_- := \sum_{\beta_j < 0} \mathbb{R} f_j$$

$$\mathfrak{e}_0 := \sum_{\beta_j = 0} \mathbb{R} f_j$$

$$\mathfrak{e}_+ := \sum_{\beta_j > 0} \mathbb{R} f_j$$

Lemma II.28

(1) $\mathfrak{e}_0 = \{X \in \mathfrak{e} : B_\mathfrak{g}(X, Y) = 0 \ \forall Y \in \mathfrak{e}\}$ = null space of $B_\mathfrak{g}$ in $\mathfrak{e}$.

(2) $[\mathfrak{e}_0, \mathfrak{e}] = 0$ and $\mathfrak{e}_0$ is an Abelian ideal in $\mathfrak{g}$

(3) $[\mathfrak{e}_-, \mathfrak{e}_+] = 0$.

Pf later

Then we define

$$\mathfrak{u}_+ := [\mathfrak{e}_+, \mathfrak{e}_+]$$

$$\mathfrak{u}_- := [\mathfrak{e}_-, \mathfrak{e}_-]$$

Lemma II.29 $\mathfrak{u}_+$ & $\mathfrak{u}_-$ are orthogonal w.r.t. $B_\mathfrak{g}$

$$B_\mathfrak{g}(X, Y) = 0 \quad \forall X \in \mathfrak{u}_+, Y \in \mathfrak{u}_-$$

Finally $\mathfrak{u}_0 := \mathfrak{u} \ominus (\mathfrak{u}_+ \oplus \mathfrak{u}_-) =$

$=$ orthogonal complement to $\mathfrak{u}_+ \oplus \mathfrak{u}_-$ in $\mathfrak{u}$ w.r.t. $B_\mathfrak{g}$.

Corollary II.30

(1) $\mathfrak{k}_0$ is an Abelian ideal in $\mathfrak{g}$

(2) $\mathfrak{u}_+ \oplus \mathfrak{k}_+$, $\mathfrak{u}_0 \oplus \mathfrak{k}_0$, $\mathfrak{u}_- \oplus \mathfrak{k}_-$ are parwise orthogonal ideal in $\mathfrak{g}$ (w.r.t. $B_\mathfrak{g}$).

We can now lemma II.28

Lemma II.28

(1) $\mathfrak{k}_0 = \{X \in \mathfrak{k} : B_\mathfrak{g}(X,Y) = 0 \ \forall Y \in \mathfrak{k}\}$ $=$ null space of $B_\mathfrak{g}$ in $\mathfrak{e}$.

(2) $[\mathfrak{k}_0, \mathfrak{k}] = 0$ and $\mathfrak{k}_0$ is an Abelian ideal in $\mathfrak{g}$

(3) $[\mathfrak{k}_-, \mathfrak{k}_+] = 0$.

Pf (1) $\mathfrak{g}^\perp := \{X \in \mathfrak{g} : B_\mathfrak{g}(X,Y) = 0 \ \forall Y \in \mathfrak{g}\}$

$B_\mathfrak{g}$ is θ-invariant $\Rightarrow$ $\mathfrak{g}^\perp$ θ-inv.

$\Rightarrow \mathfrak{g}^\perp = (\mathfrak{g}^\perp \cap \mathfrak{u}) \oplus (\mathfrak{g}^\perp \cap \mathfrak{k})$.

$(\mathfrak{g}, \theta)$ effective $\overset{\text{II.2?(2)}}{\Longrightarrow}$ $B_\mathfrak{g}|_{\mathfrak{u}\times\mathfrak{u}}$ is negative definite $\Rightarrow \mathfrak{g}^\perp \cap \mathfrak{u} = (0)$

$\Rightarrow \mathfrak{g}^\perp \subset \mathfrak{k} \Rightarrow$

$\Rightarrow \mathfrak{g}^\perp = \{X \in \mathfrak{k} : B_\mathfrak{g}(X,Y) = 0 \ \forall Y \in \mathfrak{g}\} =$

$\overset{\text{II.2?(1)}}{=} \{X \in \mathfrak{k} : B_\mathfrak{g}(X,Y) = 0 \ \forall Y \in \mathfrak{k}\}$

$= \{X \in \mathfrak{k} : \langle AX, Y \rangle = 0 \ \forall Y \in \mathfrak{k}\}$

$= \ker A = \mathfrak{k}_0$ (by defn. $\mathfrak{k}_0 = \sum_{\beta_j = 0} \mathbb{R} f_i$).

(2) $[\mathfrak{k}_0, \mathfrak{k}] \subset [\mathfrak{k}, \mathfrak{e}] \subset \mathfrak{u}$.

$X \in \mathfrak{k}_0$, $Y \in \mathfrak{k}$, $Z \in \mathfrak{u}$

$$B_\mathfrak{g}(\underset{\substack{\cap \\ \mathfrak{u}}}{[X,Y]}, \underset{\substack{\cap \\ \mathfrak{u}}}{Z}) = -B_\mathfrak{g}([Y,X], Z) =$$

$$= B_\mathfrak{g}(\underset{\substack{\cap \\ \mathfrak{k}_0}}{X}, \underset{\substack{\cap \\ \mathfrak{k}}}{[Y,Z]}) =$$

$$= \langle \underbrace{AX}_{=0}, [Y,Z] \rangle = 0$$

since $X \in \mathfrak{k}_0$

But II.2?(2) $B_\mathfrak{g}|_{\mathfrak{u}\times\mathfrak{u}} << 0$

$\Rightarrow [X,Y] = 0 \quad \forall X \in \mathfrak{k}_0$, $Y \in \mathfrak{k}$.

In particular $[\mathfrak{k}_0, \mathfrak{k}_0] = 0$

$\Rightarrow \mathfrak{k}_0$ Abelian.

$[\mathfrak{k}_0, \mathfrak{g}] = [\mathfrak{k}_0, \mathfrak{u}] + [\mathfrak{k}_0, \mathfrak{e}] =$

$= [\mathfrak{k}_0, \mathfrak{u}] \underset{\uparrow}{=} \mathfrak{k}_0$

because $\mathfrak{u}$ comm with A $\Rightarrow$ preserves $\mathfrak{k}_+, \mathfrak{k}_-, \mathfrak{k}_0$.

(3) $\forall X \in \mathfrak{k}_-$, $Y \in \mathfrak{k}_+$, $Z \in \mathfrak{u}$.

$$B_\mathfrak{g}([X,Y], Z) = -B_\mathfrak{g}(\underset{\substack{\cap \\ \mathfrak{k}_+}}{Y}, \underset{\substack{\cap \\ \mathfrak{k}_-}}{[X,Z]})$$

$$= -\langle \underset{\substack{\cap \\ \mathfrak{e}_+}}{AY}, \underset{\substack{\cap \\ \mathfrak{e}_+}}{[X,Z]} \rangle = 0$$

and $\mathfrak{e}_+ \perp \mathfrak{e}_-$ w.r.t. $\langle , \rangle$ because they are defn. using an o.n. basis.

$\Rightarrow [X,Y] = 0 \quad \forall X \in \mathfrak{k}_-$, $Y \in \mathfrak{k}_+$ $\boxed{\text{W}}$