Symmetric Spaces: Exercise Class

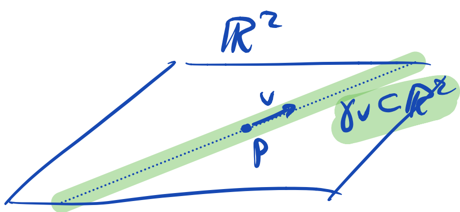
01.04.21

I) Recap:

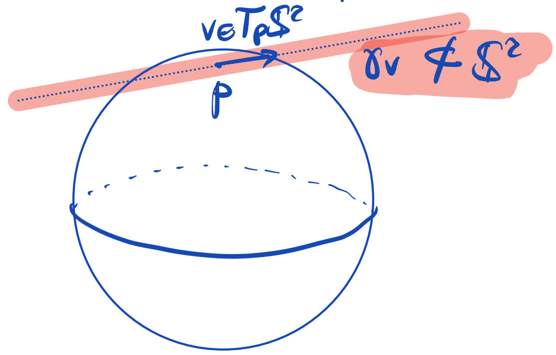
(A) Totally geodesic submanifolds of RSSs

Def: Let (M, g) be a Riem. mfld. A submfld $N \subset M$ is called <u>totally geodesic</u> if at every point $p \in N$ and for every tangent vector $v \in T_pN \subset T_pM$ the M-geodesic $\gamma_v \subset M$ starting at p in the direction of v is actually contained in N.

Ex 1: $M = \mathbb{R}^3$, $N = \mathbb{R}^2 \subset \mathbb{R}^3$ is totally geodesic



<u>Non-example:</u> $S^2 \subset \mathbb{R}^3$



Characterization for RSSs:

Def: A subspace $\mathfrak{n}$ of a Lie algebra $\mathfrak{g}$ is called a Lie triple system if $\forall X, Y, Z \in \mathfrak{n}$:

$$[X, [Y, Z]] \subset \mathfrak{n}$$

Ex 2: (G,K) Riem. sym. pair with Cartan decomp $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$.

Then $\mathfrak{p}$ is a Lie triple system:

$$[\mathfrak{p},\underbrace{[\mathfrak{p},\mathfrak{p}]}_{\subset\mathfrak{k}}]\subset[\mathfrak{p},\mathfrak{k}]\subset\mathfrak{p} \quad \checkmark$$

Thm II.25: Let $M=G/K$ RSS, $o\in M$ basept, $K=\mathrm{stab}_G(o)$, $G=\mathrm{Iso}(M)^\circ$. Cartan decomp. $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$

(1) For any Lie triple system $\mathfrak{n}\subset\mathfrak{p}$ the image $N=(\mathrm{Exp}_o\circ d_e\pi)(\mathfrak{n})\subset M$ is a totally geodesic submanifold through $o\in M$ with $T_oN=d_e\pi(\mathfrak{n})$.

$$\begin{tikzcd}
\mathfrak{n} \arrow[r] \arrow[d, phantom, "\cap"] & d_e\pi(\mathfrak{n})=T_oN \arrow[d, phantom, "\cap"] \\
\mathfrak{p} \arrow[r, "d_e\pi"] \arrow[d, "\exp"'] \arrow[dr, phantom, "//"] & T_oM \arrow[d, "\mathrm{Exp}"] \\
G \arrow[r, "\pi"] & M \supset N \\
g \arrow[r, mapsto] & g\cdot o
\end{tikzcd}$$

(2) Vice versa, if $N\subset M$ is a totally geodesic submfld through $o\in M$ then, $\mathfrak{n}:=(d_e\pi)^{-1}(T_oN)$ is a Lie triple system.

$\rightsquigarrow$ Correspondence between Lie triple systems and totally geodesic submanifolds.

(In Ex 2: $\mathfrak{p}$ corresponds to the entire symmetric space M.)

(B) Orthogonal Symmetric Lie Algebras:

Def: An orthogonal symmetric Lie algebra (OSLA) is a pair $(\mathfrak{g}, \theta)$ such that:

(1) $\mathfrak{g}$ is a real Lie algebra;

(2) $\Theta : \mathfrak{g} \to \mathfrak{g}$ is an involution such that
$$\mathfrak{u} := \{ X \in \mathfrak{g} \mid \Theta X = X \}$$
is compactly embedded
$$\mathrm{ad}(\mathfrak{u}) \subset \mathfrak{gl}(\mathfrak{g}) = \mathrm{Lie}(GL(\mathfrak{g}))$$
is the Lie algebra of a compact subgroup $\mathcal{U} < GL(\mathfrak{g})$.

Recall:

$$\begin{tikzcd}
\mathfrak{u} \subset \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"'] \arrow[dr, phantom, "="] & \mathfrak{gl}(\mathfrak{g}) \supset \mathrm{ad}(\mathfrak{u}) = \mathrm{Lie}(\mathcal{U}) \arrow[d, "\mathrm{Exp}"] \\
G \arrow[r, "\mathrm{Ad}"] & GL(\mathfrak{g}) \supset \mathcal{U}
\end{tikzcd}$$

COMPACT!

$(\mathfrak{g}, \theta)$ is called effective, if $\mathfrak{u} \cap Z(\mathfrak{g}) = 0$

center of $\mathfrak{g}$: $X \in Z(\mathfrak{g}) \Leftrightarrow [X,Y] = 0 \ \forall Y \in \mathfrak{g}$

Ex: $M = G/K$ RSS, $G = \mathrm{Isom}(M)^\circ$, $\sigma : G \to G$ involution

Set $\mathfrak{g} = \mathrm{Lie}(G)$, $\Theta := d_e\sigma$.

Then $(\mathfrak{g}, \Theta)$ is an OSLA.

Def: (Types of OSLAs)

Let $(\mathfrak{g}, \theta)$ be an effective OSLA with Killing form $B_{\mathfrak{g}}$ and decomp. $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ $\left(\begin{array}{l} \mathfrak{u} = \{X \mid \theta X = X\} \text{ "}=\mathfrak{k}\text{"} \\ \mathfrak{e} = \{X \mid \theta X = -X\} \text{ "}=\mathfrak{p}\text{"} \end{array} \right)$

(1) $(\mathfrak{g}, \theta)$ is of compact type if $B_{\mathfrak{g}}$ is negative definite;

(2) $(\mathfrak{g}, \theta)$ is of non-compact type if $B_{\mathfrak{g}}|_{\mathfrak{e}}$ is positive definite;

(3) $(\mathfrak{g}, \theta)$ is of Euclidean type if $\mathfrak{e}$ is an abelian ideal.

Thm II.27 (Decomposition thm for OSLAs)

$(\mathfrak{g}, \theta)$ effective OSLA. Then $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$ is a decomposition into θ-inv. ideals such that:

- $(\mathfrak{g}_-, \theta|_{\mathfrak{g}_-})$ is of non-compact type;
- $(\mathfrak{g}_0, \theta|_{\mathfrak{g}_0})$ is of Euclidean type;
- $(\mathfrak{g}_+, \theta|_{\mathfrak{g}_+})$ is of compact type.

Remark: This will amount later on to a decomposition of RSSs at the manifold level!

II) Exercise Sheet 2:

Exercise 1.(Invariant Riemannian metrics on homogeneous spaces):

In the first exercise class we saw that every homogeneous G-manifold M is diffeomorphic to a quotient G/H, where $H = G_p < G$ is the stabilizer subgroup of a point $p \in M$. The diffeomorphism $F\colon G/H \to M$ is given by $F(gH) = g \cdot p$. Moreover, we saw that the set $R(M)^G$ of G-invariant Riemannian metrics on M can be identified with the set $\mathrm{Sym}_+(T_pM)^H$ of H-invariant inner products on the tangent space T_pM.

Complete our discussion by showing the following:

a) Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Then the differential $dF_e\colon \mathfrak{g}/\mathfrak{h} \cong T_eG/H \to T_pM$ induces a bijection between H-invariant inner products on T_pM and $\mathrm{Ad}(H)$-invariant inner products on $\mathfrak{g}/\mathfrak{h}$.

Sol: Only need to see that $dF_e\colon \mathfrak{g}/\mathfrak{h} \longrightarrow T_pM$ is H-equivariant, i.e. $dF_e(\underbrace{Ad(h)}_{= d_e c_h} X) = dh(dF_e(X))$.

Define $\tilde{F}\colon G \to M$, $g \mapsto g\cdot p$

$$\begin{tikzcd} G \arrow[r, "\tilde F"] \arrow[d, bend right] & M \\ G/H \arrow[ur, bend right, "F"', "\cong"] & \end{tikzcd}$$

Then $\tilde{F}(\underbrace{hgh^{-1}}_{c_h(g)}) = hgh^{-1}\cdot p = hg\cdot p = h\cdot \tilde{F}(g)$

Taking derivatives:

$$d\tilde{F}_e \circ Ad(h) = dh \circ d\tilde{F}.$$

This descends to $\mathfrak{g}/\mathfrak{h} \cong T_e(G/H)$:

$$dF \circ Ad(h) = dh \circ dF.$$

$$\Longrightarrow \quad dF^*\colon \mathrm{Sym}_+(T_pM)^H \xrightarrow{\sim} \mathrm{Sym}_+(\mathfrak{g}/\mathfrak{h})^{Ad(H)}. \qquad \square$$

b) Show that every $\mathrm{Ad}(H)$-invariant inner product $\langle\cdot,\cdot\rangle \in \mathrm{Sym}_+(\mathfrak{g}/\mathfrak{h})$ is also $\mathrm{ad}(\mathfrak{h})$-invariant, i.e.

$$\langle \mathrm{ad}(X)Y, Z\rangle + \langle Y, \mathrm{ad}(X)Z\rangle = 0$$

for all $X \in \mathfrak{h}, Y, Z \in \mathfrak{g}/\mathfrak{h}$.

If H is connected, the converse holds as well: Every $\mathrm{ad}(\mathfrak{h})$-invariant inner product is $\mathrm{Ad}(H)$-invariant.

Sol: "$\mathrm{Ad}(H)$-inv. $\Rightarrow$ $\mathrm{ad}(\mathfrak{h})$-inv." By taking derivatives:

$$\langle Y, Z\rangle = \langle \mathrm{Ad}(\overbrace{\exp(t\cdot X)}^{\in H})\, Y, \mathrm{Ad}(\exp(t\cdot X))\, Z\rangle$$

$$\forall X \in \mathfrak{h} \quad \forall Y, Z \in \mathfrak{g}/\mathfrak{h} \quad \forall t \in \mathbb{R}$$

$\overset{\frac{d}{dt}|_{t=0}}{\Longrightarrow}$

$$0 = \frac{d}{dt}\Big|_{t=0} \langle \mathrm{Ad}(\exp(t\cdot X))\, Y, \underbrace{\cancel{\mathrm{Ad}(\exp(0\cdot X))\cdot Z}}_{Z}\rangle$$

$$+ \frac{d}{dt}\Big|_{t=0} \langle \overbrace{\cancel{\mathrm{Ad}(\exp(0\cdot X))}Y}^{Y}, \mathrm{Ad}(\exp(t\cdot X))\, Z\rangle$$

$$= \langle \mathrm{ad}(X)\, Y, Z\rangle + \langle Y, \mathrm{ad}(X) Z\rangle .$$

"$\Leftarrow$": The usual trick using $G = \langle U \rangle$
(Intro. to Lie Gps.)

id. neighborhood onto which exp is a diffeo.

c) Let $G = \mathrm{GL}(n,\mathbb{R})$ and let $d_1,\dots,d_m \in \mathbb{N}$ such that $d_1 + \dots + d_m = n$. Denote by $P < G$ the subgroup that consists of block upper triangular matrices of the form

$$\begin{array}{c} d_1\{ \\ d_2\{ \\ \vdots \\ d_m\{ \end{array} \begin{pmatrix} \boxed{B_1} & & * \\ & \ddots & \\ 0 & & \boxed{B_m} \end{pmatrix},$$

where $B_i \in \mathrm{GL}(d_i,\mathbb{R})$, $i = 1,\dots,m$.

Use the above characterization to show that there are no G-invariant Riemannian metrics on G/P.

Remark: The quotient space G/P can be interpreted as the flag variety of partial flags $\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_m = \mathbb{R}^n$, where $\dim V_i = d_1 + \dots + d_i$, $i = 1,\dots,m$.

Sol: Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R})$,

$$\mathfrak{p} = \mathrm{Lie}(P) = \left\{ \begin{pmatrix} \llcorner^{*} & & * \\ & \llcorner^{*} \ddots & \\ 0 & & \llcorner^{*} \end{pmatrix} \right\},$$

$$\mathfrak{l} = \left\{ \begin{pmatrix} \llcorner & & 0 \\ & \llcorner \ddots & \\ * & & \llcorner \end{pmatrix} \right\} \quad \text{"complement of } \mathfrak{p}\text{"}$$

$$\Rightarrow \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{l} \quad \Rightarrow \mathfrak{l} \cong \mathfrak{g}/\mathfrak{p}.$$

Denote
$$E_{ij} = \begin{array}{c} \\ i \rightarrow \\ \\ \end{array} \begin{array}{c} \quad\quad j \\ \begin{pmatrix} 0 & & 0 \\ & \boxed{1} & \\ 0 & & 0 \end{pmatrix} \end{array}$$

Set $H := E_{11} - E_{nn} = \begin{pmatrix} 1 & 0 \\ 0 & \boxed{-1} \end{pmatrix} \in \mathfrak{p}$

and consider $E_{n1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{l}$

Then
$$[H, E_{n1}] = [E_{11}, E_{n1}] - [E_{nn}, E_{n1}]$$
$$= -E_{n1} - E_{n1} = -2 \cdot E_{n1}$$

Suppose $\langle .,. \rangle$ is an $\mathrm{ad}(p)$-inv. inner prod.

Then:
$$0 = \langle \underbrace{[H, E_{n1}]}_{=\mathrm{ad}(H)E_{n1} = -2E_{n1}}, E_{n1} \rangle + \langle E_{n1}, \underbrace{[H, E_{n1}]}_{=-2E_{n1}} \rangle$$
$$= -4 \cdot \| E_{n1} \|^2 \neq 0. \qquad \lightning.$$

<u>Alternative Solution</u>

If G/P admits a G-inv. Riem. metric, then this will induce a G-inv. volume form $\omega_{G/P}$ on G/P.

⇝ We obtain a G-inv. Radon measure $\mu_{G/P}$ on G/P.

By "Weil's quotient measure thm" (Lie gps), we would need that
$$\Delta_G|_P \equiv \Delta_P$$

However, G is unimodular and P is not:
$$1 = \Delta_G|_P = \Delta_P \neq 1 \qquad \lightning$$

Aside on <u>flag varieties</u>:

A partial flag is a collection $(V_1, \dots, V_m)$ of nested linear subspaces $0 \subsetneq V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_m = \mathbb{R}^n$.

We fix the dimensions $\alpha_i = \dim V_i$ of the subspaces and call
$$\mathcal{F}_{\underline{\alpha}} := \{ (V_1, \dots, V_m) \mid \dim V_i = \alpha_i \}$$

a <u>flag variety</u>.

$GL_n(\mathbb{R})$ acts transitively on $\mathcal{F}_{\underline{\alpha}}$ via

$$g\cdot(V_1,\dots,V_m) = (g(V_1),\dots,g(V_m)).$$

Let $\{e_1,\dots,e_n\}$ be the standard basis of $\mathbb{R}^n$ and set

$$W_i := \langle e_1,\dots,e_{\alpha_i}\rangle \quad , i=1,\dots,m.$$

Then the stabilizer subgroup of $(W_1,\dots,W_m)$ is

$$P = \left\{ \begin{array}{r} \alpha_2 \left[\begin{array}{r} d_1\} \\ d_2\} \end{array} \right. \\ \vdots \end{array} \begin{pmatrix} \boxed{} & & & \ast \\ & \boxed{} & & \\ & & \ddots & \\ 0 & & & \boxed{} \end{pmatrix} \right\}$$

with $d_i = \alpha_i - \alpha_{i-1}$; $\alpha_0 = 0$.

Thus, we obtain a bijection

$$\begin{aligned} G/P &\xrightarrow{\sim} \mathcal{F}_{\underline{\alpha}}, \\ g\cdot P &\longmapsto g\cdot(W_1,\dots,W_m). \end{aligned}$$

Exercise 2.(Compact Lie groups as symmetric spaces):

Let G be a compact connected Lie group and let

$$G^* = \{(g,g) \in G \times G : g \in G\} < G \times G$$

denote the diagonal subgroup.

a) Show that the pair $(G \times G, G^*)$ is a Riemannian symmetric pair, and the coset space $G \times G/G^*$ is diffeomorphic to G.

Sol: We find an involutive automorphism

$$\sigma : G \times G \longrightarrow G \times G$$
$$(g_1, g_2) \longmapsto (g_2, g_1)$$

Then $G^\sigma = G^* = \{(g,g) \in G\times G \mid g \in G\} \supset (G^\sigma)^0$

(G^* is compact s.t. $\mathrm{Ad}_{G\times G}(G^*)$ is automatically compact)

$\Longrightarrow$ $(G\times G, G^*)$ is RSP.

Consider the action: $(G \times G) \times G \longrightarrow G$,

$$((g_1, g_2), g) \longmapsto g_1 g g_2^{-1}$$

and the orbit map $\varphi : G \times G \longrightarrow G$, $(g_1, g_2) \longmapsto g_1 g_2^{-1}$

$\rightsquigarrow$ $\bar{\varphi} : (G\times G)/G^* \xrightarrow{\sim} G$ is a diffeo.

b) Using the above, explain how any compact connected Lie group G can be regarded as a Riemannian globally symmetric space.

Sol: We put a $(G\times G)$-inv. Riem. metric on G.

Then $G \cong (G \times G)/G^*$ is a RGS. (see lecture).

c) Let $\mathfrak{g}$ denote the Lie algebra of G. Show that the exponential map from $\mathfrak{g}$ into the Lie group G coincides with the exponential map from $\mathfrak{g}$ into the Riemannian *globally symmetric space* G.

Sol: $\mathfrak{p} = E_{-1}(d\sigma_e)$, $d\sigma_e : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}\times\mathfrak{g}$

$(X,Y) \mapsto (Y,X)$

$(X,Y) \in \mathfrak{p} \iff (X,Y) = -(Y,X)$

$\Rightarrow \mathfrak{p} = \{(X,-X) \in \mathfrak{g}\times\mathfrak{g}\}.$

$$\begin{tikzcd}
\mathfrak{g}\times\mathfrak{g} \supseteq \mathfrak{p} \arrow[r, "d_e\pi"] \arrow[d, "\exp^* = \exp\times\exp"'] \arrow[dr, phantom, "\mathbin{/\!/\!/}"] & T_eG \arrow[d, "\mathrm{Exp}"] \\
G\times G \arrow[r, "\pi"] & G
\end{tikzcd}$$

$(g_1,g_2) \mapsto g_1 g_2^{-1}$

Notice $d\pi_e(X,Y) = X - Y$

$$\mathrm{Exp}(d\pi_e(X,-X)) = \pi(\exp^*(X,-X))$$

$$= \pi(\exp(X), \exp(-X))$$

$$= \exp(X)\cdot \underbrace{\exp(-X)^{-1}}_{=\exp(X)} = \exp(2\cdot X)$$

On the other hand: $\mathrm{Exp}(d\pi_e(X,-X)) = \mathrm{Exp}(X-(-X))$

$= \mathrm{Exp}(2X).$

Lie theoretic, Riemannian.

$\Rightarrow \exp = \mathrm{Exp}.$

$\square$

Exercise 5.(Closed differential forms):

Let M be a Riemannian globally symmetric space and let ω be a differential form on M invariant under $\mathrm{Isom}(M)^\circ$. Prove that $d\omega = 0$.

<u>Sol:</u> s_m = geodesic symmetry at $m \in M$,
$\omega \in \Omega^p(M)$ invariant.

$s_m^* \omega$ is invariant too:

$$g^* s_m^* \omega = s_m^* \underbrace{s_m^*}_{=id}\underbrace{(s_m g s_m)^*}_{\in G} \omega = s_m^* \omega$$

G acts transitively $\Rightarrow$ ω is det. by its value at a single point $m \in M$.

Recall $d_m s_m = -id$

$$(s_m^* \omega)_m (v_1, \dots, v_p) = \omega_m (ds_m(v_1), \dots, ds_m(v_p))$$
$$= \omega_m(-v_1, \dots, -v_p)$$
$$= (-1)^p \omega_m(v_1, \dots, v_p)$$

$\Rightarrow$ $s_m^* \omega = (-1)^p \omega$.

$\Rightarrow$ $d\omega = (-1)^p d(s_m^* \omega) = (-1)^p s_m^* \underbrace{(d\omega)}_{(p+1)\text{-form}} = (-1)^p (-1)^{p+1} d\omega$
$= -d\omega$

$\Rightarrow$ $d\omega = 0$.

$\square$