Lecture

14 April 2021

Goal: Decomp. of RSS by decomp. of OSLA

Defn An OSLA is a pair $(\mathfrak{g}, \Theta)$,

- $\mathfrak{g}$ real Lie alg. Θ inv., $\Theta^2 = Id$, $\Theta \neq Id$
- $\mathfrak{u} := \{X : \Theta X = X\}$ is compactly embedded ($ad(\mathfrak{u}) \subset \mathfrak{gl}(\mathfrak{g})$ is $Lie(U) = ad(\mathfrak{u})$, where $U < GL(\mathfrak{g})$ cpt)

Ex. If $M = G/K$ RSS $\Rightarrow U = Ad(K)$

We say that OSLA is effective if $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{u} = (0)$

Θ inv. $\Rightarrow \mathfrak{e} := \{X \in \mathfrak{g} : \Theta X = -X\}$

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}.$$

Ex. $M = G/K$ RSS $\Rightarrow$

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, w.r.t. Θ = Cartan involution

Cartan decomp.

Defn If $(\mathfrak{g}, \Theta)$ is an OSLA, $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ is the Cartan decomp. of $\mathfrak{g}$ w.r.t. Θ.

Lemma II.26 The decomp. is $\perp$ w.r.t. $B_{\mathfrak{g}}$.

Goal $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$

- $\mathfrak{g}_\varepsilon$ $\varepsilon \in \{-, 0, +\}$ are ideals ($B_{\mathfrak{g}_\varepsilon} = B_{\mathfrak{g}}|_{\mathfrak{g}_\varepsilon}$)
- $B_{\mathfrak{g}_-}$ is neg. defn.
- $B_{\mathfrak{g}_+}$ is pos. defn.
- $\mathfrak{e}_0$ is an Abelian ideal in $\mathfrak{g}_0$
- $(\mathfrak{g}_\varepsilon, \Theta|_{\mathfrak{g}_\varepsilon})$ effective OSLA of the appropriate type

$$\begin{pmatrix} + & \text{non-cpt type} \\ - & \text{cpt type} \\ 0 & \text{Euclidean} \end{pmatrix}$$

- The decomp. is $\perp$ w.r.t. $B_{\mathfrak{g}}$.

Strategy $\mathfrak{e}$ is $ad(\mathfrak{u})$-inv.

$\Rightarrow$ plays the role of $\mathfrak{p}$.

$\langle \, , \rangle$ U-inv. inn. prod. on $\mathfrak{e}$

$\Rightarrow \exists\, A \in End(\mathfrak{e})$ symm.

$$B_{\mathfrak{g}}(X, Y) = \langle AX, Y \rangle$$

$\forall\, X, Y \in \mathfrak{e}$

A has real e.v. $\{\beta_1, \dots, \beta_n\}$ with $\{f_1, \dots, f_n\}$ o.n. basis of eigenvectors.

$$\mathfrak{e}_\varepsilon := \sum^{\oplus}_{\varepsilon \text{ e.v.}} \mathbb{R} f_j \qquad \varepsilon \in \{-, 0, +\}$$

Lemma II.28

- $\mathfrak{e}_0$ = nullspace of $B_{\mathfrak{g}}$ is an Abelian ideal in $\mathfrak{g}$
- $[\mathfrak{e}_0, \mathfrak{e}] = 0$
- $[\mathfrak{e}_+, \mathfrak{e}_-] = 0$

Rk A commutes with $ad(\mathfrak{u})$

$\Rightarrow$ The decomp. $\mathfrak{e} = \mathfrak{e}_- \oplus \mathfrak{e}_0 \oplus \mathfrak{e}_+$ is $ad(\mathfrak{u})$-invariant.

Define $\mathfrak{u}_+ := [\mathfrak{e}_+, \mathfrak{e}_+]$

$\mathfrak{u}_- := [\mathfrak{e}_-, \mathfrak{e}_-]$

Lemma II.29 $\mathfrak{u}_+$ & $\mathfrak{u}_-$ are orth. w.r.t. $B_{\mathfrak{g}}$.

Pf $X_\varepsilon, Y_\varepsilon \in \mathfrak{e}_\varepsilon$, $\varepsilon \in \{-, +\}$

want to show that

$$B_{\mathfrak{g}}(\underbrace{[X_+,Y_+]}_{\in\,\mathfrak{u}_+}, \underbrace{[X_-,Y_-]}_{\in\,\mathfrak{u}_-}) = 0$$

$$B_{\mathfrak{g}}([X_+,Y_+],[X_-,Y_-]) \overset{\text{ad-inv.}}{=}$$

$$= -B_{\mathfrak{g}}(Y_+,[X_+,[X_-,Y_-]])$$

Jacobi

$$= B_{\mathfrak{g}}(Y_+,[X_-,\underbrace{[Y_-,X_+]}_{\in[\mathfrak{p}_-,\mathfrak{p}_+]=0}])$$

$$+ B_{\mathfrak{g}}(Y_+,[Y_-,\underbrace{[X_+,X_-]}_{\in[\mathfrak{p}_+,\mathfrak{p}_-]=0}]) \quad ■$$

Define $\mathfrak{u}_0 := \mathfrak{u} \ominus (\mathfrak{u}_+\oplus\mathfrak{u}_-)$ orth. in $\mathfrak{u}$ w.r.t. $B_{\mathfrak{g}}$.

Need properties of the $\mathfrak{u}_\varepsilon$.

Corollary II.30 $\mathfrak{u}_\varepsilon\oplus\mathfrak{p}_\varepsilon$ are pairwise orthogonal ideals, $\varepsilon\in\{-,0,+\}$

Lemma II.31

(1) $\mathfrak{u}_\varepsilon$ are ideals in $\mathfrak{u}$, p.wise orth. $B_{\mathfrak{g}}$

(2) $[\mathfrak{u}_0,\mathfrak{p}_-]=[\mathfrak{u}_0,\mathfrak{p}_+]=0$

(3) $[\mathfrak{u}_-,\mathfrak{p}_0]=[\mathfrak{u}_-,\mathfrak{p}_+]=0$

(4) $[\mathfrak{u}_+,\mathfrak{p}_0]=[\mathfrak{u}_+,\mathfrak{p}_-]=0$

Pf (1) Need to show only that they are ideals, rest is done.

- Let $\varepsilon\in\{+,-\}$.

$$[\mathfrak{u}_\varepsilon,\mathfrak{u}] \overset{\text{defn}}{=} [[\mathfrak{p}_\varepsilon,\mathfrak{p}_\varepsilon],\mathfrak{u}] \overset{\text{Jacobi}}{=}$$

$$= [[\mathfrak{p}_\varepsilon,\mathfrak{u}],\mathfrak{p}_\varepsilon] + [[\mathfrak{u},\mathfrak{p}_\varepsilon],\mathfrak{p}_\varepsilon] \overset{!}{=}$$

$\mathfrak{u}$-inv. of $\mathfrak{p}_\varepsilon$

$$= [\mathfrak{p}_\varepsilon,\mathfrak{p}_\varepsilon] + [\mathfrak{p}_\varepsilon,\mathfrak{p}_\varepsilon] = \mathfrak{u}_\varepsilon.$$

- Let $X\in\mathfrak{u}_0 \perp (\mathfrak{u}_+\oplus\mathfrak{u}_-)$, $Z\in\mathfrak{u}$ want that $[Z,X]\perp(\mathfrak{u}_+\oplus\mathfrak{u}_-)$, that is $\forall\, Y\in\mathfrak{u}_+\oplus\mathfrak{u}_-$

$$0 \overset{?}{=} B_{\mathfrak{g}}([Z,X],Y) \overset{\text{ad-inv.}}{=}$$

$$= -B_{\mathfrak{g}}(X,\underbrace{[Z,Y]}_{\in\,\mathfrak{u}_+\oplus\mathfrak{u}_-\text{ since ideals}}) = 0$$

(2) Let $Z\in\mathfrak{u}_0$, $X,Y\in\mathfrak{p}_\varepsilon$, $\varepsilon\in\{+,-\}$

$$B_{\mathfrak{g}}(\underbrace{[Z,X]}_{\in\,\mathfrak{p}_\varepsilon}, \underbrace{Y}_{\in\,\mathfrak{p}_\varepsilon}) = -B_{\mathfrak{g}}([X,Z],Y) \overset{\text{ad-inv.}}{=} B_{\mathfrak{g}}(\underbrace{Z}_{\in\,\mathfrak{u}_0},\underbrace{[X,Y]}_{\in\,\mathfrak{u}_\varepsilon}) = 0$$

ad($\mathfrak{u}$)-inv.

But $B_{\mathfrak{g}}|_{\mathfrak{p}_\varepsilon\times\mathfrak{p}_\varepsilon}$ is non-deg. $\Rightarrow$

$\Rightarrow [Z,X]=0 \quad \forall\, Z\in\mathfrak{u}_0,\ X\in\mathfrak{p}_\varepsilon.$

(3) & (4) $\varepsilon\in\{+,-\}$

$$[\mathfrak{u}_\varepsilon,\mathfrak{p}_0] \overset{\text{defn}}{=} [[\mathfrak{p}_\varepsilon,\mathfrak{p}_\varepsilon],\mathfrak{p}_0] \overset{\text{Jacobi}}{=}$$

$$= [\underbrace{[\mathfrak{p}_\varepsilon,\mathfrak{p}_0]}_{=0},\mathfrak{p}_\varepsilon] + [\underbrace{[\mathfrak{p}_0,\mathfrak{p}_\varepsilon]}_{=0},\mathfrak{p}_\varepsilon]$$

Lemma II.28

$$[\mathfrak{u}_\varepsilon,\mathfrak{p}_{-\varepsilon}] = [[\mathfrak{p}_\varepsilon,\mathfrak{p}_\varepsilon],\mathfrak{p}_{-\varepsilon}] =$$

$$= [\underbrace{[\mathfrak{p}_\varepsilon,\mathfrak{p}_{-\varepsilon}]}_{=0},\mathfrak{p}_\varepsilon] = 0$$

Lemma II.28 ■

Pf of Corollary II.30

$\mathfrak{u} = \mathfrak{u}_-\oplus\mathfrak{u}_0\oplus\mathfrak{u}_+ \quad \perp$ w.r.t. $B_{\mathfrak{g}}$

$\mathfrak{p} = \mathfrak{p}_-\oplus\mathfrak{p}_0\oplus\mathfrak{p}_+$

$\Rightarrow \mathfrak{u}_\varepsilon\oplus\mathfrak{p}_\varepsilon$ are p.wise orthogonal.

- To see that $\mathfrak{u}_0\oplus\mathfrak{p}_0 \subset \mathfrak{g}$ ideal. ($\mathfrak{g} = \mathfrak{u}\oplus\mathfrak{p}$)

We only need to see what happens for $[\mathfrak{u}_0,\mathfrak{p}]$

$$[\mathfrak{u}_0\oplus\mathfrak{p}_0,\mathfrak{u}\oplus\mathfrak{p}] =$$

$$= \underbrace{[\mathfrak{u}_0,\mathfrak{u}]}_{\subset\,\mathfrak{u}_0\subset\mathfrak{u}} + [\mathfrak{u}_0,\mathfrak{p}] + \underbrace{[\mathfrak{p}_0,\mathfrak{u}]+[\mathfrak{p}_0,\mathfrak{p}]}_{\subset\,\mathfrak{p}_0\subset\mathfrak{g}}$$

ideal Lemma II.31   ideal Lemma II.28

But $[\mathfrak{u}_0,\mathfrak{p}] = [\mathfrak{u}_0,\mathfrak{p}_0]$ Lemma II.31

But $\mathfrak{u}$ (& $\mathfrak{u}_0$ in particular) preserves the decomp of $\mathfrak{p}$ $\Rightarrow$

$\Rightarrow [\mathfrak{u}_0,\mathfrak{p}_0]\subset\mathfrak{p}_0 \Rightarrow \mathfrak{u}_0\oplus\mathfrak{p}_0$ is an ideal in $\mathfrak{g}$.

• To see that $\mathfrak{u}_\varepsilon \oplus \mathfrak{e}_\varepsilon \subset \mathfrak{g}$ is an ideal

$[\mathfrak{u}_\varepsilon \oplus \mathfrak{e}_\varepsilon, \mathfrak{u} \oplus \mathfrak{e}] =$

$= [\mathfrak{u}_\varepsilon, \mathfrak{u}] \subset \mathfrak{u}_\varepsilon$ ← lemma II.31

$+ [\mathfrak{u}_\varepsilon, \mathfrak{e}] = [\mathfrak{u}_\varepsilon, \mathfrak{e}_\varepsilon \oplus \mathfrak{e}_0 \oplus \mathfrak{e}_{-\varepsilon}] = [\mathfrak{u}_\varepsilon, \mathfrak{e}_\varepsilon]$

$+ [\mathfrak{e}_\varepsilon, \mathfrak{u}] \subset \mathfrak{e}_\varepsilon$ ($\mathfrak{u}$-inv. of dec. of $\mathfrak{e}$)

$+ [\mathfrak{e}_\varepsilon, \mathfrak{e}] = [\mathfrak{e}_\varepsilon, \mathfrak{e}_\varepsilon \oplus \mathfrak{e}_0 \oplus \mathfrak{e}_{-\varepsilon}] \subset$

$\subset [\mathfrak{e}_\varepsilon, \mathfrak{e}_\varepsilon] = \mathfrak{u}_\varepsilon$ lemma II.28

☐

Summarize $(\mathfrak{g}, \Theta)$ effective OSLA

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e} = (\mathfrak{u}_- \oplus \mathfrak{u}_0 \oplus \mathfrak{u}_+) \oplus (\mathfrak{e}_- \oplus \mathfrak{e}_0 \oplus \mathfrak{e}_+) =$$

$$= \underbrace{(\mathfrak{u}_- \oplus \mathfrak{e}_-)}_{=:\mathfrak{g}_-} \oplus \underbrace{(\mathfrak{u}_0 \oplus \mathfrak{e}_0)}_{=:\mathfrak{g}_0} \oplus \underbrace{(\mathfrak{u}_+ \oplus \mathfrak{e}_+)}_{=:\mathfrak{g}_+}$$

Does it work?

(1) $\mathfrak{g}_\varepsilon$ are p.wise orth. ideals in $\mathfrak{g}$ (w.r.t. $B_\mathfrak{g}$) (Cor. II.30).
Moreover $B_{\mathfrak{g}_\varepsilon} = B_\mathfrak{g}|_{\mathfrak{g}_\varepsilon}$.

(2) $(\mathfrak{g}, \Theta)$ effective $\Rightarrow B_\mathfrak{g}|_\mathfrak{u} \ll 0$ (lemma II.26)

(i) $B_\mathfrak{g}|_{\mathfrak{e}_-} \ll 0 \Rightarrow B_{\mathfrak{g}_-} \ll 0 \Rightarrow$
$\Rightarrow \mathfrak{g}_-$ is of compact type

(ii) $B_\mathfrak{g}|_{\mathfrak{e}_+} \gg 0 \Rightarrow \mathfrak{g}_+$ of non-cpt type

Rk In both cases $\mathfrak{g}$ is semisimple.

(3) $(\mathfrak{g}, \Theta)$ effective.
$\mathfrak{e}_0$ is an Abelian ideal, $\mathfrak{e}_0 \subset \mathfrak{g}_0$
$\Rightarrow \mathfrak{z}(\mathfrak{g}_0) = \mathfrak{z}(\mathfrak{g}) \Rightarrow$
$\Rightarrow \mathfrak{z}(\mathfrak{g}_0) \cap \mathfrak{u}_0 \subset \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{u}) = (0)$
$\Rightarrow (\mathfrak{g}_0, \Theta|_{\mathfrak{g}_0})$ is effective.

Exercise $\mathfrak{u}_\varepsilon$ is compactly embedded in $\mathfrak{g}_\varepsilon$, $\varepsilon \in \{-, 0, +\}$ (Helgason, Ch. V)

Problem

• If $\mathfrak{e}_0 = (0) \Rightarrow \Theta|_{\mathfrak{g}_0} = \Theta|_{\mathfrak{u}_0} = \mathrm{Id}$ ↯

$\Rightarrow$ the decomp.

$$\mathfrak{g} = \underbrace{(\mathfrak{u}_- \oplus \mathfrak{e}_-)}_{=:\mathfrak{g}_-} \oplus \underbrace{(\mathfrak{u}_0 \oplus \mathfrak{e}_0)}_{=:\mathfrak{g}_0} \oplus \underbrace{(\mathfrak{u}_+ \oplus \mathfrak{e}_+)}_{=:\mathfrak{g}_+}$$

is not in OSLA, since $(\mathfrak{g}_0, \Theta|_{\mathfrak{g}_0}) = (\mathfrak{u}_0, I)$ is not an OSLA.

(••) If $\mathfrak{e}_- \neq (0) \Rightarrow$
$\Rightarrow \mathfrak{g}_0 = (0)$
$\mathfrak{g}_+ = \mathfrak{u}_+ \oplus \mathfrak{e}_+$
$\mathfrak{g}_- = \mathfrak{u}_- \oplus \mathfrak{u}_0 \oplus \mathfrak{e}_-$

(••) If $\mathfrak{e}_- = (0) \Rightarrow$
$\mathfrak{g}_0 = \mathfrak{g}_- = (0)$
$\mathfrak{g}_+ = (\mathfrak{u}_+ \oplus \mathfrak{u}_0 + \mathfrak{u}_-) \oplus \mathfrak{e}_+$

(••) $\mathfrak{e}_0 = \mathfrak{e}_+ = \mathfrak{e}_-$ does not happen.

Recall $M = G/K$ RSS, $o \in M \Rightarrow$
$\Rightarrow T_oM \cong \mathfrak{p}$ in the Cartan dec.
and in general if
$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{e}$ with $\mathfrak{e} = 0$
$\Rightarrow (\mathfrak{g}, \Theta)$ is not an OSLA
because $\Theta = \mathrm{id}$.

OSLA ⇝ RSP

Recall (G, K) RSP $\Rightarrow$
$\Rightarrow (\mathfrak{g}, \Theta)$ is an OSLA with $\mathfrak{k}$ compactly embedded

$\Rightarrow \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k} = \mathrm{Lie}(Z(G) \cap K)$

Thus effective $\Leftrightarrow$ $Z(G) \cap K$ discrete

Defn A RSP (G, K) is *effective* if $Z(G) \cap K)$ is discrete

Lemma II.32 M RSS, $G = \mathrm{Iso}(M)^\circ$, $o \in M$, $K = \mathrm{Stab}_G(o)$.

If $N \trianglelefteq G$ is contained in K $\Rightarrow N = (e)$ and in particular the RSP (G, K) is effective.

Pf If $g_* o \in M \Rightarrow$

$\Rightarrow \mathrm{Stab}_G(g_* o) = gKg^{-1}$.

Since $N \trianglelefteq G$, $N < K \Rightarrow$

$$N \subset \bigcap_{g \in G} gKg^{-1} = \bigcap_{g \in G} \mathrm{Stab}(g_* o) = \{e\}$$

Since every sbgp of $Z(G)$ is normal $\Rightarrow (G, K)$ effective $\boxtimes$

Defn (1) An effective RSP (G, K) is of cpt, non-cpt, Euclidean type if the corresp. OSLA is.

(2) A RSS M is of cpt, non-cpt, Eucl. type if the corresp. RSP $(\mathrm{Iso}(M)^\circ, \mathrm{Stab}_{\mathrm{Iso}(M)^\circ}(o))$ is, $o \in M$.

Thm II.33 If M is a simply conn. RSS, then M is

$M = M_- \times M_0 \times M_+$

(Riem. product), where

M_- is of cpt type

M_0 Eucl.

M_+ is of non-cpt type.