lecture

15 April 2021

Last time: dec. of OSLA

Today: dec. of RSS (non-cpt, cpt, Eucl)

" " " (irreducible)

curvature of RSS

Thm II.33 M simply conn. RSS $\Rightarrow$

$$M \cong \underset{\text{non-cpt}}{M_+} \times \underset{\text{Eucl.}}{M_0} \times \underset{\text{cpt.}}{M_-} \quad \text{(diffeom.)}$$

Pf $G = \mathrm{Iso}(M)^\circ$, $o \in M$, $\sigma = s_o g s_o$, $\Theta = d_e\sigma \Rightarrow (\mathfrak{g}, \Theta)$ is an OSLA

$\Rightarrow \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$.

Let G_ε be the Lie sbgps of G corresp. to $\mathfrak{g}_\varepsilon$. $\mathfrak{g}_\varepsilon$ ideals $\Rightarrow G_\varepsilon$ are normal sbgps, and $G_\varepsilon \cap G_\eta$ is discrete. We claim that $[G_\varepsilon, G_\eta] = 0 \Rightarrow$

$$\varphi: G_- \times G_0 \times G_+ \longrightarrow G, \quad (x,y,z) \longmapsto xyz \text{ is a homo}$$

In fact $[G_\varepsilon, G_\eta] \underset{\text{normal}}{\subset} G_\varepsilon \cap G_\eta$, but $[G_\varepsilon, G_\eta]$ is conn. $\Rightarrow$ trivial.

$d_e\varphi: \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ \longrightarrow \mathfrak{g}$ isom.

$\Rightarrow \tilde\varphi: \tilde G_- \times \tilde G_0 \times \tilde G_+ \longrightarrow \tilde G$ isom.

Let $p: \tilde G \to G$ be the projection $\Rightarrow$

$\Rightarrow \tilde G / (p^{-1}(K))^\circ \longrightarrow \tilde G / p^{-1}(K) = G/K = M$

M simply conn $\Rightarrow p^{-1}(K) < \tilde G$ is conn.

$\mathfrak{k} = \mathrm{Lie}(p^{-1}(K)) < \mathfrak{g}$. Let $\mathfrak{k}_\varepsilon$ be the subalgebras $\mathfrak{k} = \mathfrak{k}_- \oplus \mathfrak{k}_0 \oplus \mathfrak{k}_+$ and K_ε the corresp. subgps. of $\tilde G$. $\Rightarrow \tilde\varphi(K_- \times K_0 \times K_+) = p^{-1}(K)$ and K_ε are closed. $\Rightarrow (\tilde G_\varepsilon, K_\varepsilon)$ are RSP w.r.t. the lift $\tilde\sigma$ of σ. $\Rightarrow$

$$\Rightarrow \tilde\varphi: \tilde G_-/K_- \times \tilde G_0/K_0 \times \tilde G_+/K_+ \longrightarrow M$$

diffeo. ∎

II.9 Decomposition into irreducible symmetric spaces.

Defn An OSLA $(\mathfrak{g}, \Theta)$ is *reduced* if $\mathfrak{k}$ does not contain any non-zero ideal.

Meaning any $\mathfrak{n} \subset \mathfrak{g}$ ideal, $\mathfrak{n} \subset \mathfrak{k}$ trivial $\Leftrightarrow$ any connected normal sbgp $N \lhd G$, $N < K$ is trivial.

In particular reduced $\Rightarrow$

$$\Rightarrow \bigcap_{g \in G} gKg^{-1} = \bigcap_{g \in G} \mathrm{Stab}_G(g_* o)$$

cannot be connected $\Rightarrow$ discrete

So if $(\mathfrak{g}, \Theta)$ is an OSLA $\Rightarrow G \curvearrowright G/K$ has discrete kernel.

Consequence If $(\mathfrak{g}, \Theta)$ is reduced $\Rightarrow$ effective. In fact $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k}$ is a subalg. of $\mathfrak{z}(\mathfrak{g}) \Rightarrow$ Abelian hence ideal in $\mathfrak{g}$ contained in $\mathfrak{k} \Rightarrow$ trivial.

Defn Let $(\mathfrak{g}, \Theta)$ be an OSLA with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We say that $(\mathfrak{g}, \Theta)$ is *irreducible* if

(1) $\mathfrak{g}$ semisimple & $(\mathfrak{g}, \Theta)$ reduced

(2) $\mathrm{ad}_\mathfrak{g}(\mathfrak{k})$ acts irreducibly on $\mathfrak{p}$.

$(B_\mathfrak{g}(X,Y) = \langle AX, Y \rangle \quad X, Y \in \mathfrak{p})$

Thm II.34 $(\mathfrak{g}, \Theta)$ a reduced OSLA. Then $(\mathfrak{g}, \Theta)$ is the direct sum of irreducible

OSLA and the decomp. is unique.

Thm II.35 V real v.space and $K < GL(V)$ cpt $\Rightarrow \exists$ decomposition $V = \sum^{\oplus} V_i$ into K-invariant irreducible subspaces.

Pf Let $\langle\, ,\, \rangle$ be a K-invariant inner product on V. Either done or let $W \subset V$ be a K-inv. subspace, $\dim W < \dim V$.
$\Rightarrow W^\perp$ is also invariant and $\dim W^\perp < \dim V$. $\boxtimes$

Sketch of the proof of Thm II.34

Let $\mathfrak{u}$ be as before, $\mathfrak{k}$ also and $\langle\, ,\, \rangle$ inn. prod. on $\mathfrak{e}$ that is $\mathfrak{u}$-invariant. Let $A \in \mathrm{End}(\mathfrak{e})$ be symm. s.t.

$B_{\mathfrak{g}}(X,Y) = \langle AX, Y\rangle \quad \forall X,Y \in \mathfrak{e}$

Let $\mathfrak{e} = \sum_{i=0}^{r} \mathfrak{q}_i$ be the decomposition corresponding to the distinct eigenvalues $c_0 = 0, c_1 \dots c_r \neq 0$, $c_i \neq c_j$, $1 \le i \ne j \le r$.
This decomp. is also $\mathfrak{u}$-invariant $\Rightarrow$ by Thm II.35 we can decompose the $\mathfrak{q}_i$ into $\mathfrak{u}$-inv. irred. subspaces, $\mathfrak{q}_i = \sum_{j=1}^{r_i \oplus} \mathfrak{p}_{ij}$.
The $\mathfrak{p}_{ij}$ will play the role of $\mathfrak{p}$ in the Cartan decomposition.
Define $\mathfrak{g}_{ij} := [\mathfrak{p}_{ij}, \mathfrak{p}_{ij}] + \mathfrak{p}_{ij}$
(Drop the double indices ...)

$\mathfrak{g}_i := [\mathfrak{p}_i, \mathfrak{p}_i] + \mathfrak{p}_i \quad , i = 1, \dots, n$

To show:
(1) the $\mathfrak{g}_i$ are ideals in $\mathfrak{g}$ that are Θ-invariant.
(2) $B_{\mathfrak{g}_i} = B_{\mathfrak{g}}|_{\mathfrak{g}_i \times \mathfrak{g}_i}$ non-degenerate
(3) Let $\mathfrak{m} := \sum_{i=1}^{n} \mathfrak{g}_i$: this is a semisimple Θ-inv. ideal and $\mathfrak{g}_0 := \mathrm{Centr}_{\mathfrak{g}}(\mathfrak{m})$.
This is also Θ-invariant and $(\mathfrak{g}_0, \Theta|_{\mathfrak{g}_0})$ is a Euclidean OSLA. $\boxtimes$

Details of the proof in Borel Semisimple groups & RSS Thm 1.9.

Defn A RSP (G,K) is reduced or irreducible if the corresponding OSLA is.

If M is a RSS $\Rightarrow$ the RSP $(\mathrm{Iso}(M)^\circ, \mathrm{Stab}_{\mathrm{Iso}(M)^\circ}(o))$ is reduced.

Defn A RSS M is irreducible if the corresp. RSP is.
That is if $\mathrm{Lie}(\mathrm{Iso}(M)^\circ)$ is semisimple, K acts irreducibly via Ad on $\mathfrak{p}$, when $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomp.

Corollary II.36 A RSS M is isometric to the Riem. product $M = M_0 \times \dots \times M_n$, where $M_0 = \mathbb{E}^k$, M_i, $1 \le i \le n$ are irreducible symm. spaces of compact or non-compact type.

Rk M irred. $\not\Rightarrow$ $\mathrm{Iso}(M)^\circ$ is simple. For example, let $U =$ cpt Lie gp and let us consider the RSP $(U \times U, \Delta(U))$

$\Delta(U) = \{(g,g) \in U \times U\}$,

$\Theta(x,y) = (y,x) \Rightarrow$

$\Rightarrow \mathfrak{k} = \{(X,X) : X \in \mathfrak{u}\}$

$\mathfrak{p} = \{(Y,-Y) : Y \in \mathfrak{u}\}$

with $\mathrm{ad}(\mathfrak{k})$- action

$(\mathrm{ad}(X,X))(Y,-Y) =$

$= [(X,X),(Y,-Y)] =$

$= ([X,Y], -[X,Y])$

$\Rightarrow$ If U is simple then $U \times U / \Delta(U)$ is an irreducible symmetric space.

Proposition II.37 Let (G,K) is an IRSP with Cartan dec. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and let $B_\mathfrak{g}$ be the Killing form. $\Rightarrow \exists!$ up to scalars G-inv. Riem. metric on G/K and one of the foll. holds:

(1) $B_\mathfrak{g}|_{\mathfrak{p}\times\mathfrak{p}} >> 0$ & G/K is of non-cpt. type. Riem. m. $= B_\mathfrak{g}$

(2) $B_\mathfrak{g}|_{\mathfrak{p}\times\mathfrak{p}} << 0$ & G/K is of cpt. type. Riem. m. $= -B_\mathfrak{g}$

Pf $\langle\ ,\ \rangle$ $\mathrm{Ad}(K)$-inv. inner product on $\mathfrak{p}$,

$B_\mathfrak{g}(X,Y) = \langle AX, Y \rangle$, $\forall X,Y \in \mathfrak{p}$

Since $\mathrm{Ad}(K)$ acts irred. on $\mathfrak{p}$ $\Rightarrow A = \lambda \, \mathrm{Id}_\mathfrak{p}$ for $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Whether $B_\mathfrak{g}$ is pos. defn. or neg. defn. depends on the sign of λ. $\square$

II.10 Curvature

M Riem. mfd with Levi-Civita connection ∇.

Defn The curvature of M is a multilinear map

$R : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to \mathrm{Vect}(M)$

(where $\mathrm{Vect}(M)$ is a $C^\infty(M)$-module) defn as follows

$R(X,Y)Z := \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z$

Rk $(R(X,Y)Z)_p$ depends only on X_p, Y_p, Z_p.

From R one can define the sectional curvature:

Let $\mathrm{Gr}_2(T_pM)$ be the Grassmannian of 2-planes in T_pM.

$K_p : \mathrm{Gr}_2(T_pM) \longrightarrow \mathbb{R}$

$P \longmapsto \langle R(u,v)u, v \rangle$

where $\{u,v\}$ is an orthonormal basis of P.

Thm II.38 (G,K) RSP with associated sym. space M with G-invariant Riem. metric.

(1) (G,K) cpt type $\Rightarrow K_p \geq 0$ $\forall p \in M$

(2) (G,K) non-cpt type $\Rightarrow K_p \leq 0$

(3) (G,K) Euclidean $\Rightarrow K_p \equiv 0$.

The proof relies on:

Thm II.39 (G,K) RSP, M RSS and curvature R. Then at $o \in M$

$$\left(R(\bar{X}_1,\bar{X}_2)\bar{X}_3\right)_o = -\overline{[[X_1,X_2],X_3]},$$

where $\bar{X}_i := d_e\pi(X_i)$, $X_i \in \mathfrak{p}$, $i=1,2,3$.

(Helgason, Thm IV 4.2 p.215)

Pf of Thm II.38 $X_1,X_2 \in \mathfrak{p}$

$$B_{\mathfrak{g}}(-[[X_1,X_2],X_1],X_2) \overset{*}{=}$$
$$= -B_{\mathfrak{g}}([X_1,X_2],[X_1,X_2])$$

(1) If (G,K) is of compact type

$\Rightarrow$ take $-B_{\mathfrak{g}}$ as Riem. metric at o after $\mathfrak{p} \simeq T_oM$.

Let $X_1,X_2 \in \mathfrak{p}$ s.t. $\bar{X}_1,\bar{X}_2 \in T_oM$ are orthonormal.

$$K_o(\mathrm{span}(\bar{X}_1,\bar{X}_2)) =$$
$$\overset{\text{defn.}}{=} -\langle R(\bar{X}_1,\bar{X}_2)\bar{X}_1,\bar{X}_2\rangle$$
$$\overset{\text{II.39}}{=} -\langle \overline{[[X_1,X_2],X_1]},\bar{X}_2\rangle$$
$$= -B_{\mathfrak{g}}([[X_1,X_2],X_1],X_2) =$$
$$= B_{\mathfrak{g}}([X_1,X_2],[X_1,X_2]) =$$
$$= \langle [X_1,X_2],[X_1,X_2]\rangle =$$
$$= \|[X_1,X_2]\|^2 \geq 0.$$