lecture

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II.11 Duality

- V $\mathbb{R}$-vector space: a **complex structure** on V is $J \in \mathrm{End}(V)$, $J^2 = -\mathrm{Id}$
  $(\alpha + i\beta)v := \alpha v + \beta J(v) \rightsquigarrow \tilde{V}$ $\mathbb{C}$-v.s.
- E $\mathbb{C}$-vectorspace $\rightsquigarrow E^{\mathbb{R}}$ $\mathbb{R}$-v.s. & $\widetilde{(E^{\mathbb{R}})} = E$
- $\mathfrak{v}$ $\mathbb{R}$-Lie algebra: a **complex structure** on $\mathfrak{v}$ is a $J \in \mathrm{End}(\mathfrak{v})$, $J^2 = -\mathrm{Id}$ s.t. $[X, JY] = J[X,Y]$ $\forall X,Y \in \mathfrak{v}$ $\Rightarrow [\,,\,] : \tilde{\mathfrak{v}} \times \tilde{\mathfrak{v}} \to \tilde{\mathfrak{v}}$ is $\mathbb{C}$-bilinear & $\tilde{\mathfrak{v}}$ is a $\mathbb{C}$-Lie alg.

W $\mathbb{R}$-vector space, $J \in \mathrm{End}(W \times W)$, $J(v,w) = (-w, v)$. Set $W^{\mathbb{C}} = \widetilde{W \times W}$ = **complexification** of W. On $W^{\mathbb{C}}$ define the **complex conjug.**
$\tau \in \mathrm{End}(W \times W)$, $\tau(v,w) = (v,-w)$

$W \hookrightarrow W^{\mathbb{C}}$
$v \longmapsto (v,0)$ as an $\mathbb{R}$-subspace.

$\Rightarrow$ • $W^{\mathbb{C}} \cong W + iW$ $\quad (X,Y) \longmapsto X + iY$

• $Z \in W^{\mathbb{C}}$ can be written ! $Z = X + iY$, $X,Y \in W$

• $\tau(Z) = X - iY$.

If $\mathfrak{g}$ is a $\mathbb{R}$-Lie algebra the bracket on $\mathfrak{g}$ extends to a bracket on $\mathfrak{g}^{\mathbb{C}}$.

Ex $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R}) \Rightarrow \mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C})$.
In fact $X \in \mathfrak{sl}(n,\mathbb{C}) \Leftrightarrow \mathrm{tr}\, X = 0$
$\Leftrightarrow \mathrm{Re}\,\mathrm{tr}\, X = \mathrm{Im}\,\mathrm{tr}\, X = 0 \Leftrightarrow$
$\Leftrightarrow \mathrm{tr}(\mathrm{Re}\, X) = \mathrm{tr}(\mathrm{Im}\, X) = 0 \Leftrightarrow$
$\Leftrightarrow X = X_1 + iX_2$, $X_i \in SL(n,\mathbb{R})$
$\mathfrak{sl}(n,\mathbb{C}) = \mathfrak{sl}(n,\mathbb{R}) + i\,\mathfrak{sl}(n,\mathbb{R})$.

Ex $\mathfrak{g} = \mathfrak{su}(n,\mathbb{C}) = \{X \in \mathfrak{sl}(n,\mathbb{C}) : X^* + X = 0\}$, where $X^* = \bar{X}^t$. $\mathfrak{su}(n,\mathbb{C})$ is only a $\mathbb{R}$-algebra (if $X \in \mathfrak{su}(n,\mathbb{C}) \Rightarrow$
$\Rightarrow \alpha X \notin \mathfrak{su}(n,\mathbb{C})$, $\alpha \in \mathbb{C}$:
$X^* + X = 0 \Rightarrow \bar{\alpha} X^* + \alpha X \neq 0$)

$\mathfrak{su}(n,\mathbb{C}) = \{X \in \mathfrak{sl}(n,\mathbb{C}) : X^* + X = 0\}$
$\mathfrak{su}(n,\mathbb{C})^{\mathbb{C}} = \mathfrak{sl}(n,\mathbb{C})$. In fact
$i\,\mathfrak{su}(n,\mathbb{C}) = \{iX \in \mathfrak{sl}(n,\mathbb{C}) : X^* + X = 0\}$
$= \{Y \in \mathfrak{sl}(n,\mathbb{C}) : Y^* = Y\}$

If $A \in \mathfrak{sl}(n,\mathbb{C}) \Rightarrow$

$$A = \underbrace{\frac{A - A^*}{2}}_{\in\, \mathfrak{su}(n,\mathbb{C})} + \underbrace{\frac{A + A^*}{2}}_{\in\, i\,\mathfrak{su}(n,\mathbb{C})}$$

$\Rightarrow \mathfrak{sl}(n,\mathbb{C}) \subseteq \mathfrak{su}(n,\mathbb{C}) + i\,\mathfrak{su}(n,\mathbb{C})$
and a count of dimensions gives equality.

Ex $\mathfrak{g} = \mathfrak{o}(p,q)$ = Lie algebra of $O(p,q)$ = orth. gp. of a non-deg. quadr. form of sign. (p,q). However all these q.f are equiv. over $\mathbb{C}$

$\Rightarrow \mathfrak{o}(p,q)^{\mathbb{C}} = \mathfrak{o}(p+q,\mathbb{C})$.
In particular $\mathfrak{o}(n,\mathbb{R})^{\mathbb{C}} = \mathfrak{o}(n,\mathbb{C})$

Defn. Let $\mathfrak{h}$ be a $\mathbb{C}$-Lie algebra.
$\Rightarrow \mathfrak{g}$ is a **real form** of $\mathfrak{h}$ if $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}$.
If $\mathfrak{g}$ is semisimple & $B_{\mathfrak{g}}$ is neg. def. $\Rightarrow \mathfrak{g}$ is a **compact form** of $\mathfrak{h}$.
Sometimes: a cpt form of a $\mathbb{R}$-Lie alg is a cpt form of its complexification.

Thm II.41 (Helgason Thm III.6.3 p.181)
Every $\mathbb{C}$-Lie algebra has a compact form.

Lemma II.40 $\mathfrak{g}$ $\mathbb{R}$-Lie algebra
(1) $B_{\mathfrak{g}}(X,Y) = B_{\mathfrak{g}^{\mathbb{C}}}(X,Y)$ $\forall X,Y \in \mathfrak{g}$
(2) $B_{(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}}}(X,Y) = 2\,\mathrm{Re}\, B_{\mathfrak{g}^{\mathbb{C}}}(X,Y)$
$X, Y \in \mathfrak{g}^{\mathbb{C}}$

(3) $\mathfrak{g}$ semisimple $\Leftrightarrow$ $\mathfrak{g}^{\mathbb{C}}$ s.s. $\Leftrightarrow$ $(\mathfrak{g}^{\mathbb{C}})^{\mathbb{R}}$ s.s.

Let $(\mathfrak{g},\Theta)$ be an OSLA with Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. Let $\mathfrak{g}^* := \mathfrak{k}+i\mathfrak{p}$ be a subspace of $\mathfrak{g}^{\mathbb{C}}$. Since $[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k},i\mathfrak{p}]\subset i\mathfrak{p}$, $[i\mathfrak{p},i\mathfrak{p}]\subset\mathfrak{k}$
$\Rightarrow$ $\mathfrak{g}^*$ is a Lie algebra with bracket coming from $\mathfrak{g}^{\mathbb{C}}$.
$\tau:\mathfrak{g}^{\mathbb{C}}\to\mathfrak{g}^{\mathbb{C}}$ conj. $\tau|_{\mathfrak{g}^*}\in \mathrm{End}(\mathfrak{g}^*)$ is involutive. $\tau(X+iY)=X-iY$.

Proposition II.42 Let $(\mathfrak{g},\Theta)$ be an OSLA with $\mathfrak{g}$ semisimple.

(1) $(\mathfrak{g}^*,\Theta^*)$ is an OSLA, where $\Theta^*=\tau|_{\mathfrak{g}^*}$

(2) $(\mathfrak{g}^*)^{\mathbb{C}}=\mathfrak{g}^{\mathbb{C}}$, $(\Theta^*)^{\mathbb{C}}=\Theta^{\mathbb{C}}$

(3) $(\mathfrak{g}^*,\Theta^*)$ effective $\Leftrightarrow$ $(\mathfrak{g},\Theta)$ effective

(4) $(\mathfrak{g}^*,\Theta^*)$ reduced $\Leftrightarrow$ $(\mathfrak{g},\Theta)$ reduced

(5) $(\mathfrak{g}^*,\Theta^*)$ is of cpt type (non-cpt type) $\Leftrightarrow$ $(\mathfrak{g},\Theta)$ is of non-compact type (cpt type)

(6) $(\mathfrak{g}_1,\Theta_1)\sim(\mathfrak{g}_2,\Theta_2)$ $\Leftrightarrow$
$\Leftrightarrow$ $(\mathfrak{g}_1^*,\Theta_1^*)\sim(\mathfrak{g}_2^*,\Theta_2^*)$

(7) $((\mathfrak{g}^*)^*,(\Theta^*)^*)=(\mathfrak{g},\Theta)$

Defn $(\mathfrak{g}_1,\Theta_1)\sim(\mathfrak{g}_2,\Theta_2)$ if $\exists\ \varphi:\mathfrak{g}_1\to\mathfrak{g}_2$ Lie alg. isom. s.t. $\Theta_2\circ\varphi=\varphi\circ\Theta_1$

Pf (1) $\mathfrak{k}$ is the fixed pt algebra of Θ^*, so we only need to show that $\mathfrak{k}$ is cptly emb. in $\mathfrak{g}^*$.
Consider the v.s. isom $\varphi:\mathfrak{g}\to\mathfrak{g}^*$
$X+Y\mapsto X+iY$ $\Rightarrow$ $\Phi:GL(\mathfrak{g})\to GL(\mathfrak{g}^*)$
$A\longmapsto\varphi A\varphi^{-1}$.
$\mathfrak{k}$ cptly emb. in $\mathfrak{g}$ $\Rightarrow$ $\exists\ U\subset GL(\mathfrak{g})$ cpt. conn. s.t. $\mathrm{Lie}(U)=\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})$.
Using Φ and its derivative
$d_I\Phi:\mathfrak{gl}(\mathfrak{g})\to\mathfrak{gl}(\mathfrak{g}^*)$ $\Rightarrow$
$\Rightarrow$ $\Phi(U)$ is a cpt. conn. Lie gp $\Phi(U)<GL(\mathfrak{g}^*)$ s.t.
$\mathrm{Lie}\,\Phi(U)=\mathrm{ad}_{\mathfrak{g}^*}(\mathfrak{k})$.

(2) Verification

(3) $\mathfrak{z}(\mathfrak{g})\cap\mathfrak{k}=\mathfrak{z}(\mathfrak{g}^*)\cap\mathfrak{k}$

(4) $(\mathfrak{g},\Theta)$ reduced $\Leftrightarrow$
$\Leftrightarrow$ any ideal $\mathfrak{r}$ in $\mathfrak{g}$ that is cont. in $\mathfrak{k}$ is trivial
$\Leftrightarrow$ $\mathfrak{r}\subset\mathfrak{k}$ is an ideal and $[\mathfrak{r},\mathfrak{p}]=0$

$\Leftrightarrow$ $\mathfrak{r}\subset\mathfrak{k}$ is an ideal and $[\mathfrak{r},i\mathfrak{p}]=0$
$\Leftrightarrow$ any ideal in $\mathfrak{g}^*$ that is cont. in $\mathfrak{k}$ is trivial $\Leftrightarrow$
$\Leftrightarrow$ $(\mathfrak{g}^*,\Theta^*)$ is reduced

(5) Need to look at the sign of the Killing form restricted to the complement of $\mathfrak{k}$.
Lemma II.40 $\Rightarrow$ $\forall\ X,Y\in\mathfrak{p}$ $\Rightarrow$
$B_{\mathfrak{g}}(X,Y)=B_{\mathfrak{g}^{\mathbb{C}}}(X,Y)=-B_{\mathfrak{g}^{\mathbb{C}}}(iX,iY)$
$=-B_{\mathfrak{g}^*}(iX,iY)$ (use (2))

(6) Any isom. of real Lie algebras extends to an isom. of the complexifications.

Defn We call $(\mathfrak{g}^*,\Theta^*)$ the *dual* of $(\mathfrak{g},\Theta)$.

Ex $(\mathfrak{sl}(n,\mathbb{R}),\Theta)$, $\Theta(X)=-X^t$.
$\mathfrak{sl}(n,\mathbb{C})=\mathfrak{sl}(n,\mathbb{R})^{\mathbb{C}}$. The dual of $(\mathfrak{sl}(n,\mathbb{R}),\Theta)$ is $(\mathfrak{su}(n),\Theta^*)$.
In fact if $\mathfrak{sl}(n,\mathbb{R})=\mathfrak{k}\oplus\mathfrak{p}$
$\mathfrak{k}=\{X\in\mathfrak{sl}(n,\mathbb{R}):X+X^t=0\}$
$\mathfrak{p}=\{X\in\mathfrak{sl}(n,\mathbb{R}):X=X^t\}\Rightarrow$

$$\Rightarrow \mathfrak{g}^*=\mathfrak{k}+i\mathfrak{p}=$$
$$=\{Z\in\mathfrak{sl}(n,\mathbb{C}): Z=X+iY \text{ where } X+X^t=0,\ Y=Y^t\}$$
$$=\{Z\in\mathfrak{sl}(n,\mathbb{C}): Z+Z^*=0\}=$$
$$=\mathfrak{su}(n)$$

The corresp. sym. spaces are
$M=SL(n,\mathbb{R})/SO(n)$, $M^*=SU(n)/SO(n)$
where M^* is compact.

Ex $\mathfrak{g}=\mathfrak{so}(n)=\{X\in\mathfrak{gl}(n,\mathbb{R}):X+X^t=0\}$ and let $p,q\in\mathbb{N}\cup\{0\}$ be such that $p+q=n$. Define $\Theta:\mathfrak{gl}(n,\mathbb{R})\to\mathfrak{gl}(n,\mathbb{R})$
$\Theta(X):=I_{p,q}XI_{p,q}$, $I_{p,q}=\begin{pmatrix}-I_p&0\\0&I_q\end{pmatrix}$
Easy to see that Θ preserves $\mathfrak{g}$ and $\Theta^2=-\mathrm{Id}$. Write also $\mathfrak{g}$ in block form

$$\mathfrak{g}=\left\{\begin{pmatrix}A&B\\-B^t&D\end{pmatrix}: \begin{matrix}A+A^t=0\\D+D^t=0\end{matrix},\ B\in M_{p,q}(\mathbb{R})\right\}$$

$$\Theta(X)=\Theta\begin{pmatrix}A&B\\-B^t&D\end{pmatrix}=\begin{pmatrix}A&-B\\B^t&D\end{pmatrix}\Rightarrow$$

$$\mathfrak{k}=\{X\in\mathfrak{so}(n,\mathbb{R}):\Theta(X)=X\}=$$
$$=\left\{\begin{pmatrix}A&0\\0&D\end{pmatrix}: \begin{matrix}A\in\mathfrak{so}(p,\mathbb{R})\\D\in\mathfrak{so}(q,\mathbb{R})\end{matrix}\right\}\ \&$$

$$\mathfrak{p}=\{X\in\mathfrak{so}(n,\mathbb{R}):\Theta(X)=-X\}=$$
$$=\left\{\begin{pmatrix}0&B\\-B^t&0\end{pmatrix}: B\in M_{p,q}(\mathbb{R})\right\}.$$

$\Rightarrow \mathfrak{g}^*\subset\mathfrak{gl}(n,\mathbb{C})$

$$\mathfrak{k}+i\mathfrak{p}=\left\{\begin{pmatrix}A&iB\\-iB^t&D\end{pmatrix}: \begin{matrix}A+A^t=0\\B\in M_{p,q}(\mathbb{R})\\D+D^t=0\end{matrix}\right\}$$

To identify $\mathfrak{g}^*$, define

$$\sigma:\mathfrak{gl}(n,\mathbb{C})\to\mathfrak{gl}(n,\mathbb{C})$$
$$Y\mapsto\begin{pmatrix}-iI_p&0\\0&I_q\end{pmatrix}Y\begin{pmatrix}iI_p&0\\0&I_p\end{pmatrix}$$

$$\sigma\begin{pmatrix}A&iB\\iB^t&D\end{pmatrix}=\begin{pmatrix}A&B\\B^t&D\end{pmatrix}\Rightarrow$$

$\Rightarrow\sigma$ isom. of $\mathfrak{g}^*$ with

$$\mathfrak{so}(p,q)=\left\{\begin{pmatrix}A&B\\B^t&D\end{pmatrix}: \begin{matrix}A+A^t=0\\B\in M_{p,q}(\mathbb{R})\\D+D^t=0\end{matrix}\right\}$$

$$\Theta^*=\tau|_{\mathfrak{g}^*},\quad \tau(X+iY)=X-iY$$
$$\Rightarrow\Theta^*\begin{pmatrix}A&B\\B^t&D\end{pmatrix}=\begin{pmatrix}A&-B\\-B^t&D\end{pmatrix}$$

$\Rightarrow(\mathfrak{so}(p+q,\mathbb{R}),\Theta)$ and $(\mathfrak{so}(p,q),\Theta^*)$ are dual OSLA.
The corresp. RSS are
$M^*=SO(p,q)/SO(p)\times SO(q)$ ~~not cpt~~
$M=SO(p+q,\mathbb{R})/SO(p)\times SO(q)$ cpt

Thm II.43 Let $M=G/K$ be a RSS of non-cpt type with dual $M^*=G^*/K$. Then there is a canonical isomorphism

$$\Omega^k(M)^G\simeq H^k(M^*,\mathbb{R}),$$

where $\Omega^k(M)^G$ = G-invariant smooth differential forms on M
$H^k(M^*,\mathbb{R})$ = singular cohomology of M^* with $\mathbb{R}$-coeff.

If V is a $\mathbb{R}$-v.s. $\Rightarrow$
$\mathrm{Alt}_k(V)$ = alternating forms $V^k \to \mathbb{R}$

Lemma II.44 M RSS, $G = \mathrm{Iso}(M)^0$, $o \in M$, $K = \mathrm{Stab}_G(o)$, $\pi: G \to G/K$

$d_e\pi : \mathfrak{p} \to T_oM$ isom. of v.s. that commutes with the actions of K on $\mathfrak{p}$ via $\mathrm{Ad}_\mathfrak{g}(K)$ and on T_oM via the diff. of left translation. Then

$$\Omega^k(M)^G \longrightarrow \mathrm{Alt}_k(\mathfrak{p})^{\mathrm{Ad}(K)}$$

is an isomorphism.

Pf Let $\omega \in \Omega^k(M)^G$, $\omega_o \in$
$\in \mathrm{Alt}_k(T_oM) \rightsquigarrow (d_e\pi)^*(\omega_o) \in$
$\in \mathrm{Alt}_k(\mathfrak{p})$ ------ ∎

Lemma II.45 (Cartan)

M RSS, $G = \mathrm{Iso}(M)^0$, $\omega \in \Omega^k(M)^G$.
$\Rightarrow d\omega = 0$

Pf $o \in M$, $s_o \in G$ geod. sym. at o
$\omega \in \Omega^k(M)^G$. Since $\forall g \in G \to$
$\Rightarrow s_o g s_o \in G \Rightarrow (s_o)^*\omega \in \Omega^k(M)^G$
$(\omega = (s_o g s_o)^*\omega = s_o^* g^* s_o^* \omega \Rightarrow$
$\Rightarrow s_o^*\omega = g^* s_o^* \omega)$.

Moreover $s_o|_{T_oM} = -\mathrm{Id} \Rightarrow$
$\Rightarrow (s_o^*\omega)_o = (-1)^k \omega_o$ and by G-invariance $(s_o^*\omega)_x = (-1)^k \omega_x$ $\forall x \in M$. Thus

$$\left.\begin{array}{l} d(s_o^*\omega) = (-1)^k d\omega \\ \quad \| \\ s_o^*(d\omega) = (-1)^{k+1} d\omega \end{array}\right\} d\omega = 0.$$

∎