Ex. Class: Symmetric Spaces 22.04.2021

I) Recap:

A) Decomposition of RSSs:

Thm II.33 If M is a simply conn. RSS, then M is

$$M = M_- \times M_0 \times M_+$$

(Riem. product), where

M_- is of cpt type

M_0 eucl.

M_+ is of non-cpt type.

We can decompose these factors further into irreducible factors:

Defn A RSS M is irreducible if the corresp. RSP is.
That is if Lie $(\mathrm{Iso}(M)^\circ)$ is semisimple, K acts irreducibly via Ad on $\mathfrak{p}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomp.

Corollary II.36 A RSS M is isometric to the Riem. product $M = M_0 \times \dots \times M_n$, where $M_0 = \mathbb{E}^k$, M_i, $1 \le i \le n$ are irreducible symm. spaces of compact or non-compact type.

B) Curvature of RSSs:

Thm II.39: (G, K) RSP, $M = G/K$, $K = \mathrm{stab}(o)$

then

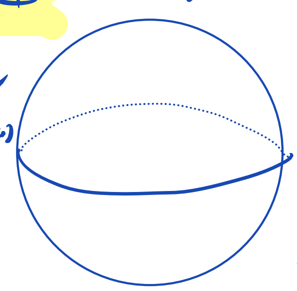
$$(R(\overline{X}_1, \overline{X}_2)\overline{X}_3)_o = -\overline{[[X_1, X_2], X_3]}$$

where we write $\overline{X}_i = d_e\pi(X_i)$

for $X_i \in \mathfrak{p} \overset{d_e\pi}{\cong} T_o M$, $\pi : G \to G/K \cong M$

S^2 sphere

$SO(n+1)/SO(n)$



$SO(1,n)^0/SO(n)$

Thm II.38 (G, K) RSP with associated symm. space M with G-invariant Riem. metric.

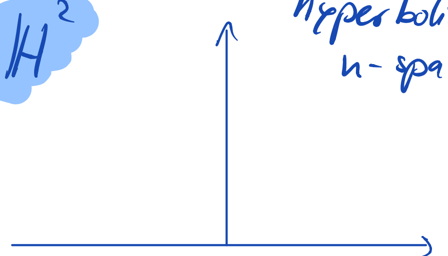
(1) (G, K) cpt type $\Rightarrow K_p \geq 0$ $\forall p \in M$

(2) (G, K) non-cpt type $\Rightarrow K_p \leq 0$

(3) (G, K) Euclidean $\Rightarrow K_p \equiv 0$.

$\mathbb{H}^2$ hyperbolic n-space

$\mathbb{R}^n$



C) Duality:

Given an OSLA $(\mathfrak{g}, \Theta)$ we can consider the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomp. of $(\mathfrak{g}, \Theta)$.

Then the dual OSLA of $\mathfrak{g}$ is given by $(\mathfrak{g}^*, \Theta^*)$ where

$$\mathfrak{g}^* := \mathfrak{k} \oplus i \cdot \mathfrak{p} \subseteq \mathfrak{g}^{\mathbb{C}} \quad \text{and}$$

$\Theta^* := \tau|_{\mathfrak{g}^*}$ is complex conjugation:

$$\tau(X + i \cdot Y) = X - i \cdot Y.$$

Important Property: Dual OSLAs are of "opposite types":

$(\mathfrak{g}, \Theta)$ compact type

$\Rightarrow (\mathfrak{g}^*, \Theta^*)$ non-compact type

& $(\mathfrak{g}, \Theta)$ non-compact type

$\Rightarrow (\mathfrak{g}^*, \Theta^*)$ compact type

Ex: $\mathbb{S}^n = SO(n+1)/SO(n)$ and $\mathbb{H}^n = SO(1,n)^0/SO(n)$ are dual to each other; see Sheet 4 Ex 4!

II) Exercise Sheet 3

Exercise 1.(Details on $\mathrm{SO}(1,n)^\circ/\mathrm{SO}(n)$**):** $\mathbb{H}^n$

Consider $G = \mathrm{SO}(1,n)^\circ$ with the involutive Lie group automorphism

$$\sigma : G \to G, g \mapsto J_n g J_n$$

where

$$J_n = \begin{pmatrix} -1 & 0 \\ 0 & I_n \end{pmatrix} \in \mathrm{SO}(1,n).$$

Further let

$$K = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SO}(n) \end{pmatrix} \cong \mathrm{SO}(n).$$

It can be shown that (G, K, σ) is a Riemannian symmetric pair and that G/K is isometric to $\mathbb{H}^n$.

a) Show that $\Theta = d\sigma : \mathfrak{g} \to \mathfrak{g}$ takes the form

$$\Theta(X) = \begin{pmatrix} 0 & -x^t \\ -x & D \end{pmatrix}$$

for all

$$X = \begin{pmatrix} 0 & x^t \\ x & D \end{pmatrix} \in \mathfrak{g} = \mathfrak{so}(1,n).$$

Deduce that

$$\mathfrak{p} = E_{-1}(\Theta) = \left\{ \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix} : x \in \mathbb{R}^n \right\}, \mathfrak{k} = E_1(\Theta) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} : D \in \mathfrak{so}(n) \right\} \cong \mathfrak{so}(n).$$

Sol:

$$\Theta(X) = d\sigma(X) = \frac{d}{dt}\Big|_{t=0} J_n \exp(t \cdot X) J_n$$

$$= J_n X J_n$$

$$= \begin{pmatrix} 0 & -x^t \\ -x & D \end{pmatrix} \quad \text{for } X = \begin{pmatrix} 0 & x^t \\ x & D \end{pmatrix} \in \mathfrak{g}.$$

$$\Theta(X) = X \iff X = \begin{pmatrix} 0 & -x^t \\ -x & D \end{pmatrix} = \begin{pmatrix} 0 & x^t \\ x & D \end{pmatrix}$$

$$\Leftrightarrow \quad X = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \,\middle|\, D \in \mathfrak{so}(n) \right\}$$

$$\cong \mathfrak{so}(n).$$

$$\theta(X) = -X \quad \Leftrightarrow \quad \begin{pmatrix} 0 & -x^t \\ -x & D \end{pmatrix} = \begin{pmatrix} 0 & -x^t \\ -x & -D \end{pmatrix}$$

$$\Leftrightarrow \quad D = 0 \quad \text{so} \quad X = \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix}.$$

b) Let $\pi : G \to G/K$ denote the usual quotient map and set $\overline{X} := d_e\pi(X) \in T_o(G/K)$ for all $X \in \mathfrak{g}$. Further let $\langle X, Y \rangle := \frac{1}{2}\operatorname{tr}(XY)$ for all $X, Y \in \mathfrak{p}$.
Show that

$$R_o(\overline{X}, \overline{Y})\overline{Z} = \langle X, Z \rangle \overline{Y} - \langle Y, Z \rangle \overline{X}$$

for all $X, Y, Z \in \mathfrak{p}$. Deduce that G/K has constant sectional curvature -1.
<u>Hint:</u> You may use the following formula without proof:
The Riemann curvature tensor at $o \in M = G/K$ is given by

$$R_o(\overline{X}, \overline{Y})\overline{Z} = -\overline{[[X, Y], Z]}$$

for all $\overline{X}, \overline{Y}, \overline{Z} \in T_oM$. (This is Thm II.39 from above)

<u>Sol:</u> Let $X = \begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & y^t \\ y & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 0 & z^t \\ z & 0 \end{pmatrix} \in \mathfrak{p}.$

Notice $\langle X, Y \rangle = \frac{1}{2} \operatorname{tr}\left(\begin{pmatrix} 0 & x^t \\ x & 0 \end{pmatrix} \begin{pmatrix} 0 & y^t \\ y & 0 \end{pmatrix} \right) = \frac{1}{2} \operatorname{tr} \begin{pmatrix} x^t y & 0 \\ 0 & x y^t \end{pmatrix}$

$= x^t y = \langle x, y \rangle_{\mathbb{R}^n}$ $\left[x y^t = \begin{pmatrix} x_1 y_1 & & * \\ & x_2 y_2 & \\ * & & \ddots \; x_n y_n \end{pmatrix} \quad \operatorname{tr}(x y^t) = x^t y. \right]$

Direct computation yields:

$$[X,Y] = \begin{pmatrix} 0 & 0 \\ 0 & xy^t - yx^t \end{pmatrix}$$

$$[[X,Y],Z] = \begin{pmatrix} 0 & \langle y,z\rangle x^t - \langle z,x\rangle \cdot y^t \\ \langle y,z\rangle x - \langle x,z\rangle y & 0 \end{pmatrix}$$

$$= \langle Y,Z\rangle X - \langle X,Z\rangle \cdot Y$$

$$R_0(\bar{X},\bar{Y})\bar{Z} = \langle Y,Z\rangle \bar{X} - \langle X,Z\rangle \cdot \bar{Y}.$$

Sectional Curvature:

Let $V \subseteq \mathfrak{p}$ be a 2D subspace. Pick an ONB $\{X,Y\}$.
Then the sectional curv. of $\bar{V} \subseteq T_0 H^n$ is:

$$K_0(\bar{V}) = \langle R_0(\bar{X},\bar{Y})\bar{Y}, \bar{X}\rangle_0$$

$$= \langle \overbrace{\cancel{\langle X,Y\rangle}}^{0} \bar{Y} - \langle Y,Y\rangle \bar{X}, \bar{X}\rangle_0$$

$$= -\underbrace{\langle Y,Y\rangle}_{=1} \underbrace{\langle \bar{X},\bar{X}\rangle_0}_{=\langle X,X\rangle = 1} = -1.$$

c) Compute that

$$\exp\left(t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

for all $t \in \mathbb{R}$.

Sol: $\exp\left(t\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}^n}_{= \begin{cases}\begin{pmatrix}0&1\\1&0\end{pmatrix}, & \text{if } n=2k+1\\ \begin{pmatrix}1&0\\0&1\end{pmatrix}, & \text{if } n=2k\end{cases}}$

$$= \underbrace{\left(\sum_{k=0}^{\infty}\frac{t^{2k+1}}{(2k+1)!}\right)}_{=\sinh(t)}\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix} + \underbrace{\left(\sum_{k=0}^{\infty}\frac{t^{2k}}{(2k)!}\right)}_{=\cosh(t)}\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

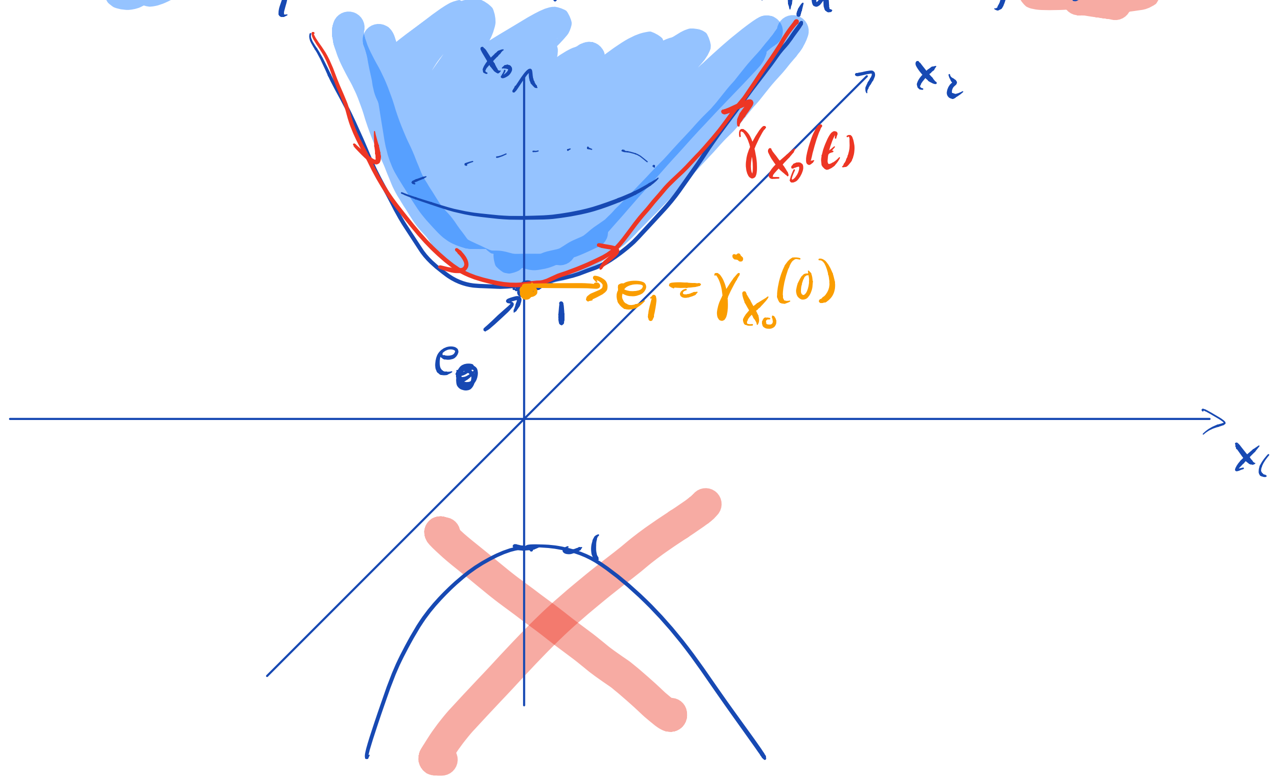
$$= \begin{pmatrix}\cosh(t) & \sinh(t)\\ \sinh(t) & \cosh(t)\end{pmatrix}. \quad \forall t\in\mathbb{R}. \quad \checkmark$$

Excursion: Hyperboloid model for $\mathbb{H}^n$

Consider the space $\mathbb{R}^{1,n} \cong \mathbb{R}^{n+1}$ endowed with the non-deg. bil. form:

$$\langle x,y\rangle_{1,n} = -x_0\cdot y_0 + x_1\cdot y_1 + \dots + x_n\cdot y_n .$$

Set $\mathcal{H}^n := \{ x\in\mathbb{R}^{1,n} \mid \langle x,x\rangle_{1,n} = -1,\ x_0>0\}.$



$\rightsquigarrow$ $SO(1,n)^\circ$ acts transitively on $\mathcal{H}^n$ via the linear action $SO(1,n) \times \mathcal{H}^n \longrightarrow \mathcal{H}^n$, $(g,x) \longmapsto g \cdot x$

The stabilizer of $e_0 = (1,0,\dots,0) \in \mathcal{H}^n$ is given by

$$SO(n) \cong \left(\begin{array}{c|c} 1 & \\ \hline & SO(n) \end{array}\right) =: K \leq SO(1,n)^\circ .$$

Cont. $\rightsquigarrow$ rk. thm

$$\varphi : SO(1,n)^\circ / K \xrightarrow[\text{diffeo.}]{\sim} \mathcal{H}^n$$

$$g \cdot K \longmapsto g \cdot e_0$$

The geodesics through e_0 are of the form

$$\gamma_X(t) := \exp(t \cdot X) \cdot e_0 \quad \text{for } X \in \mathfrak{p}.$$

For $X_0 = \left(\begin{array}{c|c} 0 & 1\ 0 \cdots 0 \\ \hline 1 & \\ 0 & 0 \\ \vdots & \\ 0 & \end{array}\right) = \left(\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right) \in \mathfrak{p}$:

$$\gamma_{X_0}(t) = \exp\left(\begin{array}{c|c} t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \\ \hline 0 & 0 \end{array}\right) \cdot e_0 = \left(\begin{array}{c|c} \exp\left(t \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) & 0 \\ \hline 0 & I_n \end{array}\right) \cdot e_0$$

$$= \left(\begin{array}{c|c} \begin{matrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{matrix} & 0 \\ \hline 0 & I_n \end{array}\right) \cdot e_0 = \begin{pmatrix} \cosh(t) \\ \sinh(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

& $\dot{\gamma}_{X_0}(0) = e_1$

We can compute any other geodesic through e_0

by using the adjoint repr. of $K = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & SO(n) \end{array}\right)$:

$$\overset{\mathrm{Ad}(K)}{\circlearrowright}\ \mathfrak{p} \xrightarrow[\sim]{d\varphi \circ d\pi} T_{e_0}\mathcal{H}^n \ \overset{K}{\underset{\text{linear}}{\circlearrowleft}}$$

For any $v \in T_{e_0}\mathcal{H}^n \cong \{0\} \times \mathbb{R}^n$ there is $k \in K$ s.t. $k \cdot e_1 = v$. Then the geodesic in the direction of v through e_0 is given by

$$\exp(t \cdot \mathrm{Ad}(k)(X_1)) \cdot e_0 = k \cdot \begin{pmatrix} \cosh(t) \\ \sinh(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

Exercise 2.(Closed adjoint subgroups of $\mathrm{SL}_n(\mathbb{R})$ and their symmetric spaces):

Consider the Riemannian symmetric pair (G, K, σ) where $G = \mathrm{SL}_n(\mathbb{R})$, $K = \mathrm{SO}(n, \mathbb{R})$ and $\sigma : \mathrm{SL}_n(\mathbb{R}) \to \mathrm{SL}_n(\mathbb{R}), g \mapsto (g^{-1})^t$. Further let $H \leq G$ be a closed, connected subgroup that is adjoint, i.e. it is closed under transposition $h \mapsto h^t$.

a) Show that $(H, H \cap K, \sigma|_H)$ is again a Riemannian symmetric pair.

Sol: Note that $\sigma(h) = (h^{-1})^t \in H$ $\forall h \in H$.

$\rightsquigarrow$ σ restricts to an invol. Lie gp. autom. $\sigma|_H : H \to H$.

Need to check:

(a) $(H^\sigma)^0 \subseteq H \cap K \subseteq H^\sigma$, and

(b) $Ad(H \cap K) \leq Gl(\mathfrak{h})$ is cpt.

(a): Notice: $(H^\sigma)^0 \subseteq H^\sigma \leq G^\sigma$ whence

$(H^\sigma)^0 \leq (G^\sigma)^0 \subseteq K$ s.t. $(H^\sigma)^0 \subseteq H \cap K$.

Moreover, $H^\sigma = H \cap G^\sigma$ and $K \subseteq G^\sigma$

$\implies$ $H \cap K \subseteq H \cap G^\sigma = H^\sigma$. ✓

(b): K is cpt., H is closed $\Rightarrow$ $H \cap K$ cpt.

$\xRightarrow[\text{cont.}]{Ad}$ $Ad(H \cap K) \leq GL(\mathfrak{h})$ is cpt.

✓

b) Show that $i : H \hookrightarrow G$ descends to a smooth embedding $\phi : H/H \cap K \hookrightarrow G/K$ such that its image is a totally geodesic submanifold of G/K.

Sol:

$$\begin{tikzcd} H \arrow[r, hook, "i"] \arrow[d, "\bar{\pi}"] \arrow[dr, phantom, "="] & G \arrow[d, "\pi"] \\ H/H\cap K \arrow[r, dashed, "\varphi"] & G/K \supseteq N \end{tikzcd}$$

- φ is well-def. & injective ✓
- φ is smooth: By defn. of smooth st. of $H/H\cap K$.
- φ is a smooth immersion:

  H-equivariance:

  $$H \curvearrowright H/H\cap K \xrightarrow{\varphi} G/K \curvearrowleft H$$

  $\left(\varphi(h \cdot \bar{\pi}(h_0)) = h \cdot \varphi(\bar{\pi}(h_0))\right)$

  $H \curvearrowright H/H\cap K$ transitive $\Longrightarrow$ φ has const. rank.

- φ is a homeo. ($\Rightarrow$ φ is a smooth embedding):

  WTS: φ is proper, i.e. $\varphi^{-1}(C)$ cpt. for $C \subset G/K$ cpt.

$$\begin{tikzcd} i^{-1}(\pi^{-1}(C)) = \pi^{-1}(C) \cap H \subseteq H \arrow[r, "i"] \arrow[d, "\bar{\pi}"] \arrow[dr, phantom, "="] & G \supseteq \pi^{-1}(C) \text{ is again cpt.} \arrow[d, "\pi \text{ has cpt. fibres!}"] \\ \varphi^{-1}(C) = \underbrace{\bar{\pi}(\pi^{-1}(C) \cap H)}_{\text{cpt.}} \subseteq H/H\cap K \arrow[r, "\varphi"] & G/K \underset{\text{cpt}}{\supseteq} C \end{tikzcd}$$

- $N = \varphi(H/H\cap K) \subseteq G/K$ is a totally geodesic submanifold:

<u>WTS</u>: $\mathfrak{n} := d_e\pi^{-1}(T_o N)$ is a Lie triple system.

Observe that the Cartan decomp. for $\mathfrak{h}$ wrt. $\overline{\Theta} := d_e\sigma|_H$ is given by

$\mathfrak{h} = \mathfrak{p}' \oplus \mathfrak{k}'$ where $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{h}$, $\mathfrak{k}' = \mathfrak{k} \cap \mathfrak{h}$.

and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ (Cartan decomp. for $d\sigma$).

<u>Then</u>: $\mathfrak{n} = (d_e\pi)^{-1}(T_o N) = (d_e\pi^{-1})(d\varphi(d\overline{\pi}_e(\mathfrak{p}')))$

$= d_e i(\mathfrak{p}') \subseteq \mathfrak{p}$

Because $d_e i$ is a Lie alg. hom. ($i : H \hookrightarrow G$ is a Lie gp. hom) and $\mathfrak{p}'$ is a Lie triple system

$\Rightarrow$ $di_e(\mathfrak{p}') = \mathfrak{n}$ is a Lie triple system:

$$[[\mathfrak{n},\mathfrak{n}],\mathfrak{n}] = di(\underbrace{[[\mathfrak{p}',\mathfrak{p}'],\mathfrak{p}']}_{\subseteq \mathfrak{p}'}) \subseteq \mathfrak{n}.$$ ✓