Lecture

28 April 2021

Thm II.43 $M = G/K$ RSS of non-cpt type, $M^* = G^*/K$ cpt. dual $\Rightarrow \exists$ can. isom.

$$\Omega^k(M)^G \simeq H^k(M^*, \mathbb{R})$$

Lemma II.44 M RSS, $G = \mathrm{Iso}(M)^\circ$, $o \in M$, $d_e\pi : \mathfrak{p} \to T_oM \Rightarrow$

$$\Rightarrow \Omega^k(M)^G \xrightarrow{\simeq} \mathrm{Alt}_k(\mathfrak{p})^{\mathrm{Ad}(K)}.$$

Lemma II.45 (Cartan) M RSS $\Rightarrow$ every G-inv. diff. form on M is closed.

Lemma II.46 U cpt. gp acting on a smooth mfd X. The inclusion of complexes

$$\Omega^k(X)^U \longrightarrow \Omega^k(X)$$

induces an isom. in cohomology

$$\Omega^k(X)^U \simeq H^k_{dR}(X).$$

Pf U cpt $\Rightarrow \mu$ norm. Haar m

Inject.: let $\alpha \in \Omega^k(X)^U$ exact in $\Omega^k(X)$, $\alpha = d\beta$, $\beta \in \Omega^{k-1}(X)$

want to show that $\alpha = d \ldots$ with $\ldots \in \Omega^{k-1}(X)^U$.

$$\alpha = u^*\alpha = u^*d\beta = d(u^*\beta) = \int_U d(u^*\beta)\, d\mu(u) = d\Big(\underbrace{\int_U u^*\beta\, d\mu(u)}_{\in\, \Omega^{k-1}(X)^U}\Big).$$

Surject. let $\alpha \in \Omega^k(U)$, $d\alpha = 0$.

U connected $\Rightarrow$ any $u \in U$ is diffeotopic to $I \in U \Rightarrow [\alpha] = [u^*\alpha] \in H^k_{dR}(X) \Rightarrow \forall$ cycle $z \in H_k(X, \mathbb{R})$

$$\int_z \alpha = \int_z u^*\alpha \quad \forall\, u \in U \Rightarrow$$

$$\Rightarrow \int_z \alpha = \int_z u^*\alpha = \int_U \Big(\int_z u^*\alpha\Big) d\mu(u) \overset{\text{Fubini}}{=} \int_z \int_U u^*\alpha\, d\mu(u) \Rightarrow$$

$$\Rightarrow \int_z \Big(\alpha - \int_U u^*\alpha\, d\mu(u)\Big) = 0 \quad \forall z \in H_k$$

$\overset{\text{deRham}}{\Rightarrow}$ $\alpha - \int_U u^*\alpha\, d\mu(u)$ repres. the zero class and in particular

$$\alpha = \int_U u^*\alpha\, d\mu(u) \text{ is } U\text{-inv.}$$

Pf of Thm II.43 want to show that $\Omega^k(M)^G \simeq H^k(M^*, \mathbb{R})$

$$\Omega^k(M)^G \underset{\text{II.44}}{\simeq} \mathrm{Alt}(\mathfrak{p})^{\mathrm{Ad}(K)} \simeq$$

$$\simeq \mathrm{Alt}(i\mathfrak{p})^{\mathrm{Ad}(K)} \underset{\text{II.44}}{=} \Omega^k(M^*)^{G^*} \underset{\substack{\text{II.46} \\ G^* \text{ cpt}}}{=}$$

$$\simeq H^k_{dR}(M^*, \mathbb{R}) \overset{\text{deRham}}{=} H^k(M^*, \mathbb{R}). \quad \blacksquare$$

III. Symmetric spaces of non-cpt type

III.1 Symm. spaces are CAT(0)

CAT = Cartan-Alexandrov-Toponogov (Gromov, 1987)

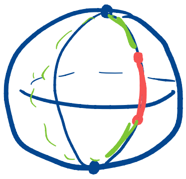
(X,d) is geodesic if $\forall\, x, y \in X$ $\exists$ cont. path $\gamma : [0, d(x,y)] \to X$ that joins x & y and $\ell(\gamma) = d(x,y)$.

If $\sigma : [a,b] \to X$ path

$$\ell(\sigma) = \sup\Big\{ \sum_{j=0}^{n-1} d(\sigma(t_j), \sigma(t_{j+1})),\ t_0 = a \le t_1 \le \ldots \le t_n = b \Big\}.$$

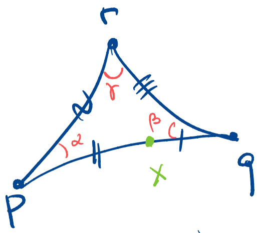
Examples are Riem. mfd, simpl. trees, buildings...

A geodesic triangle in a g.m.s. X consists of $p, q, r \in X$ and geod. segments $[p,q]$, $[q,r]$, $[p,r]$ that join them whose length is the distance between the endpts.

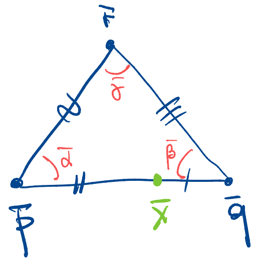


Given $\Delta(p,q,r)$, a comparison triangle $\bar\Delta$ is a triangle
$$\bar\Delta = \bar\Delta(\bar p,\bar q,\bar r) \subset \mathbb{E}^2$$
whose sides are geod. segments of the same length as the sides in Δ.

Fact: Compar. triangles always exist and are unique up to isometries.



$\Delta(p,q,r)$



$\Rightarrow d_X(p,q) = d_{\mathbb{E}^2}(\bar p,\bar q)$,
$d_X(p,r) = d_{\mathbb{E}^2}(\bar p,\bar r)$
$d_X(q,r) = d_{\mathbb{E}^2}(\bar q,\bar r)$.

Given a pt $x\in[p,q]$, a comparison pt for it is a pt $\bar x\in[\bar p,\bar q]$ s.t.
$$d_X(p,x) = d_{\mathbb{E}^2}(\bar p,\bar x)$$

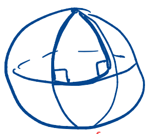
Defn. A geod. m. sp. is CAT(0) if $\forall\ \Delta(p,q,r)$ with comparison triangle $\bar\Delta(\bar p,\bar q,\bar r)$ and pts $x,y\in\Delta(p,q,r)$ with comp. pts $\bar x,\bar y\in\bar\Delta$ the inequality
$$d_X(x,y) \le d_{\mathbb{E}^2}(\bar x,\bar y)$$

In a CAT(0) space triangles are thin, that is $\alpha\le\bar\alpha$, $\beta\le\bar\beta$, $\gamma\le\bar\gamma$.

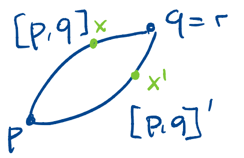
Bridson-Haefliger
"Metric spaces of non-positive curvature".

CAT(κ) spaces: triangles in CAT(κ) are thinner than triangle in a model space of const. curvature κ.

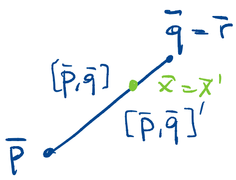
$\rightsquigarrow$ CAT(−1), CAT(0), CAT(1)
hyp. sp., eucl. pl., sphere.



Rk CAT(0) spaces are uniquely geodesic.



$\Delta(p,q,q)$



$\bar\Delta(\bar p,\bar q,\bar q)$

$$d_X(x,x') \le d_{\mathbb{E}^2}(\bar x,\bar x') = 0$$

Proposition II.1 (X,d) complete CAT(0) space.

(1) If $S\subset X$ is a bdd set and
$r_S := \inf\{r>0 : S\subset \bar B(x,r) \text{ for some } x\in X\}$
$\Rightarrow \exists!\ x_S\in X$ s.t. $S\subset \bar B(x_S,r_S)$
(circumcenter)

(2) Let $C\subset X$ be a closed convex set. Then $\exists!\ p_C(x)\in C$ that is
$d(x,p_C(x)) \le d(x,C) =$
$= \inf\{d(x,y) : y\in C\}$.

(3) Let $\gamma_1,\gamma_2 : \mathbb{R}\to X$ geod. param. by arclength. The map
$$\mathbb{R}\to\mathbb{R},\ t\longmapsto d(\gamma_1(t),\gamma_2(t))$$
is convex.

Pf (1) Let $(r_n)_{n\in\mathbb{N}}\subset\mathbb{R}$ be a sequence $r_n\to r_S$, that has the property that $\exists x_n\in X$ with $S\subset \bar B(x_n,r_n)$.

Claim (x_n) is Cauchy.

If so, let $x_S := \lim x_n \Rightarrow$

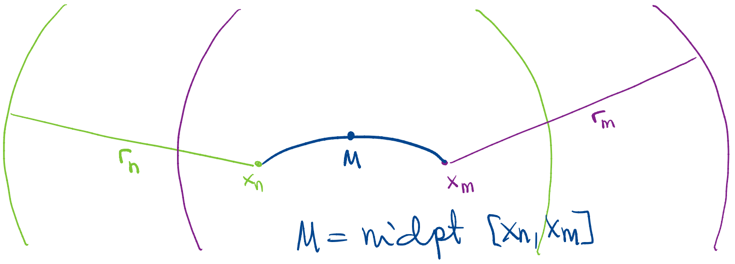
$\Rightarrow S \subset \bar{B}(x_S, r_S)$.

To see it is unique, let x_S, x'_S be two such pts, define

$$y_n := \begin{cases} x_S & n \text{ even} \\ x'_S & n \text{ odd} \end{cases}$$

$\Rightarrow S \subset \bar{B}(x_S, r_S) \cap \bar{B}(x'_S, r_S)$

$\Rightarrow S \subset \bar{B}(y_n, r_S)$ the sequence (y_n) is Cauchy $\Rightarrow x_S = x'_S$.

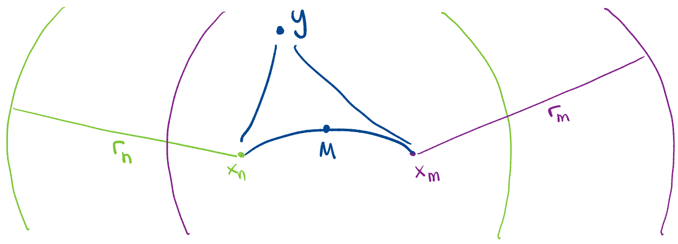


$M = $ midpt $[x_n, x_m]$

$S \subset \bar{B}(x_n, r_n)$ $\quad$ $S \subset \bar{B}(x_m, r_m)$

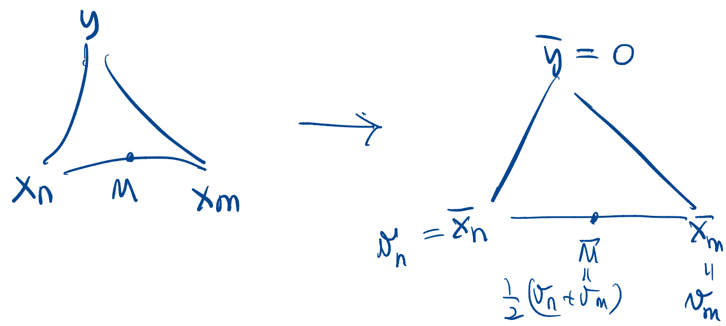
let $\varepsilon > 0$. Then $\exists\, y \in S$ s.t. $d(y,M)^2 > r_S^2 - \varepsilon$. (by defn of r_S)

(If not $\Rightarrow \forall y \in S$ $d(y,M)^2 \leq r_S^2 - \varepsilon$ $\Rightarrow S \subset \bar{B}(M, r_S^2 - \varepsilon)$ ↯)



$\Delta = \Delta(x_n, x_m, y) \rightsquigarrow \bar{\Delta}(\bar{x}_n, \bar{x}_m, \bar{y})$

$M \in [x_n, x_m] \rightsquigarrow \bar{M} \in [\bar{x}_n, \bar{x}_m]$



$$\left\| \frac{v_n + v_m}{2} \right\|^2 = d(\bar{y}, \bar{M})^2 \overset{CAT(0)}{\geq}$$

$$\geq d(y, M)^2 > r_S^2 - \varepsilon. \Rightarrow$$

$$\Rightarrow -2\langle v_n, v_m \rangle \overset{(*)}{<} \|v_n\|^2 + \|v_m\|^2 - 4(r_S^2 - \varepsilon)$$

But

$$\|v_n - v_m\| = \|v_n\|^2 + \|v_m\|^2 - 2\langle v_n, v_m \rangle \overset{(*)}{<} 2\|v_n\|^2 + 2\|v_m\|^2 - 4(r_S^2 - \varepsilon).$$

By defn. of r_S, given the chosen $\varepsilon > 0$ $\exists\, N \in \mathbb{N}$ s.t. $r_k^2 < r_S^2 - \varepsilon$ $\forall k \geq N$.

We hence choose $n, m \geq N \Rightarrow$

$\Rightarrow \|v_n\|^2 = d(\bar{x}_n, \bar{y})^2 < r_S^2 - \varepsilon$

$\|v_m\|^2 = d(\bar{x}_m, \bar{y})^2 < r_S^2 - \varepsilon$

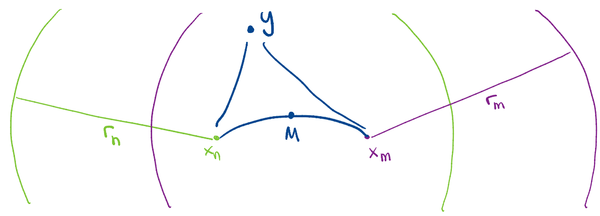
$\Rightarrow \|v_n - v_m\|^2 \leq 8\varepsilon$.

$\Rightarrow d_X(x_n, x_m) \leq 8\varepsilon$.

by dl

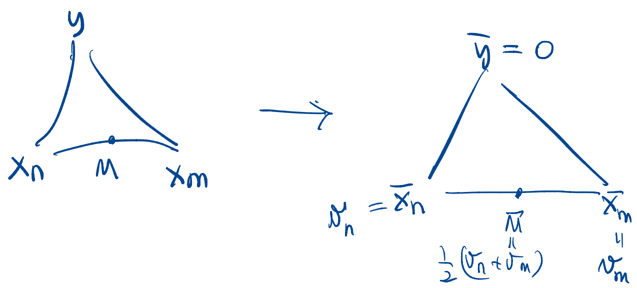
let $\varepsilon > 0$. Then $\exists\, y \in S$ s.t. $d(y,M)^2 > r_S^2 - \varepsilon$.

(If not $\Rightarrow \forall y \in S$ $d(y,M)^2 \leq r_S^2 - \varepsilon$ $\Rightarrow S \subset B(M, r_S^2 - \varepsilon)$ ↯)



$\Delta = \Delta(x_n, x_m, y) \rightsquigarrow \bar{\Delta}(\bar{x}_n, \bar{x}_m, \bar{y})$

$M \in [x_n, x_m] \rightsquigarrow \bar{M} \in [\bar{x}_n, \bar{x}_m]$



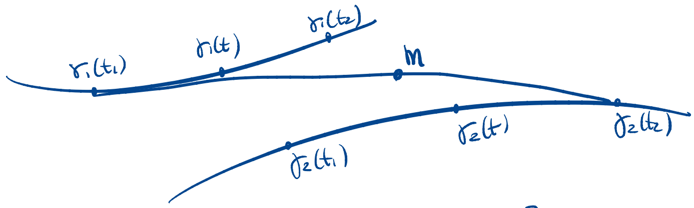
(2) Exercise

(3) $f(t) := d(\gamma_1(t), \gamma_2(t))$

convex ?

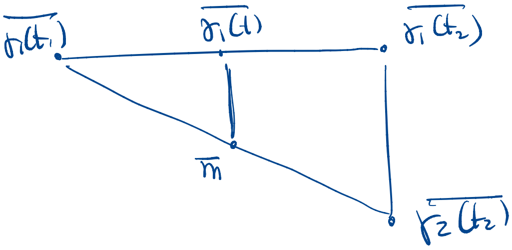
t_1, t_2, $t := \frac{1}{2}(t_1+t_2)$: want

$f(t) \le \frac{1}{2}\left(f(t_1)+f(t_2)\right)$.



$m =$ midpt $[\gamma_1(t_1), \gamma_2(t_2)]$

Consider the comp. triangle

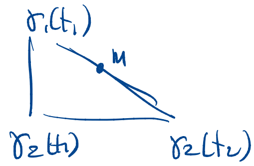


$\Rightarrow d_X(\gamma_1(t), m) \le d(\overline{\gamma_1(t)}, \overline{m}) =$

$= \frac{1}{2} d\left(\overline{\gamma_1(t_2)}, \overline{\gamma_2(t_2)}\right) =$

$= \frac{1}{2} d_X(\gamma_1(t_2), \gamma_2(t_2))$

likewise



$d_X(\gamma_2(t), m) \le$

$\le \frac{1}{2} d_X(\gamma_1(t_1), \gamma_2(t_2))$

$\Rightarrow d_X(\gamma_1(t), \gamma_2(t)) =$

$= d_X(\gamma_1(t), m) + d_X(m, \gamma_2(t))$

$\le \frac{1}{2}\Big(d_X(\gamma_1(t_1), \gamma_2(t_1)) +$

$+ d_X(\gamma_1(t_2), \gamma_2(t_2))\Big)$. $\blacksquare$