lecture

09 April 2021

Theorem III.3 RSS of non-cpt type are CAT(0).

Proposition III.3 M RSS of non-cpt type with $B_\mathfrak{g}$. Then $\mathrm{Exp}_o : T_oM \to M$ is a distance increasing diffeo and hence M is uniquely geodesic.

Pf We want to show $X \in T_oM$

$$\| d_X \mathrm{Exp}_o(\xi) \| \geq \|\xi\| \quad \forall X \in T_X(T_oM) \cong T_oM.$$

In fact $d_X(\mathrm{Exp}_o \circ d_e\pi)$ $X \in \mathfrak{p}$.

$$d_X(\mathrm{Exp}_o \circ d_e\pi) = (d_o L_{\exp X} \circ d_e\pi) \cdot \left(\sum_{n=0}^{\infty} \frac{(T_X|_\mathfrak{p})^n}{(2n+1)!} \right),$$

where $T_X = (\mathrm{ad}_\mathfrak{g} X)^2$.

Because of the metric $\Rightarrow$

- $d_e\pi$ preserves the scalar product.
- $d_o L_{\exp X} : T_oM \to T_{(\exp X)_* o}M$ is parallel transport along $t \longmapsto (\exp tX)_* o$ $\Rightarrow$ it preserves the scalar product

So let $T := \sum_{n=0}^{\infty} \frac{(T_X|_\mathfrak{p})^n}{(2n+1)!}$

Use the following:

(1) $X, Y, Z \in \mathfrak{p} \Rightarrow B_\mathfrak{g}(T_XY, Z) = B_\mathfrak{g}(Y, T_XZ)$

(2) $X, Y \in \mathfrak{p} \Rightarrow B_\mathfrak{g}(T_XY, Y) =$

$= B_\mathfrak{g}(\mathrm{ad}_\mathfrak{g}(X)\, Y, Y) =$

$= -B_\mathfrak{g}(\mathrm{ad}_\mathfrak{g}(X)\, Y, \mathrm{ad}_\mathfrak{g}(X) Y) \geq 0$

because $\mathrm{ad}_\mathfrak{g}(X)\, Y \in \mathfrak{k}$ and $B_\mathfrak{g}|_{\mathfrak{k}\times\mathfrak{k}} \ll 0$.

$\Rightarrow \exists$ o.n. basis $\{e_1, \dots, e_n\}$ of $\mathfrak{p}$ s.t. $T_X = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, $\lambda_j > 0$

$\Rightarrow T = \mathrm{diag}(\mu_1, \dots, \mu_n)$ with

$$\mu_j = \sum_{k=0}^{\infty} \frac{\lambda_j^k}{(2k+1)!} \geq 1 \Rightarrow$$

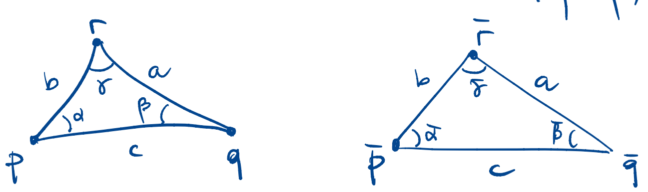
$\Rightarrow T$ is distance increasing. ☒

Rk Thm $\forall$ complete Riem. mfd of non-positive curvature.

Corollary III.4 (1) (Law of cosines) M RSS non-cpt. type, Δ geod. triangle with a, b, c as sides, and opposite angles α, β, γ. Then

$$c^2 \geq a^2 + b^2 - 2ab\cos\gamma$$

(2) If $\Delta = \Delta(p, q, r)$ with $\bar\Delta = \bar\Delta(\bar p, \bar q, \bar r) \Rightarrow \alpha \leq \bar\alpha,\ \beta \leq \bar\beta,\ \gamma \leq \bar\gamma$



(1) $v_r, w_r \in T_rM$ with $\|v_r\| = a$, $\|w_r\| = b$ $\Rightarrow$

$\Rightarrow \|v_r - w_r\|^2 = a^2 + b^2 - 2ab\cos\gamma$

Exponentiating $q = \mathrm{Exp}(v_r)$, $p = \mathrm{Exp}(w_r)$.

Angle between $[p, r]$ & $[q, r]$ is by defn. γ. $\overset{\text{Prop. III.3}}{\Rightarrow}$

$$\mathrm{length}([p,q])^2 = d(\mathrm{Exp}(v_r), \mathrm{Exp}(w_r)) \geq d(v_r, w_r) = a^2 + b^2 - 2ab\cos\gamma.$$

(2) From (1)

$$\left.\begin{array}{l} c^2 = a^2 + b^2 - 2ab\cos\bar\gamma \\ c^2 \geq a^2 + b^2 - 2ab\cos\gamma \end{array}\right\} \Rightarrow$$

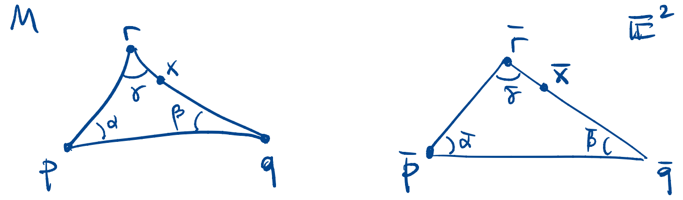
$\Rightarrow \gamma \leq \bar\gamma$. ☒

Pf of Thm III.2

Two steps:

(1) $x \in [q, r]$ with $\bar x \in [\bar q, \bar r] \Rightarrow$

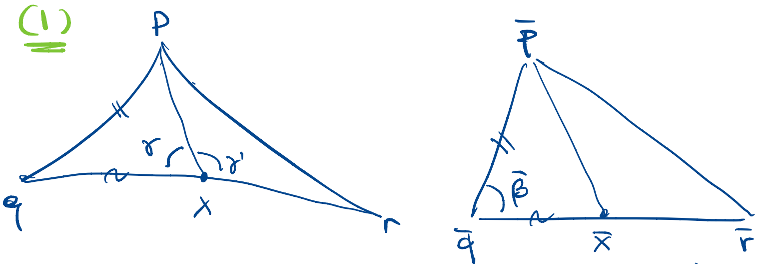
$\Rightarrow d_M(p, x) \leq d_{\mathbb{E}^2}(\bar p, \bar x)$



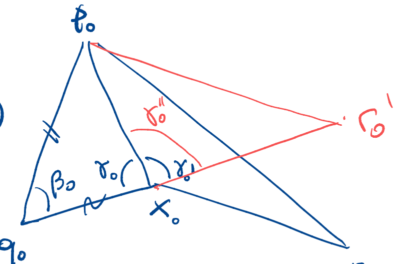
2) $\forall\, x,y \in \Delta$ with $\bar{x},\bar{y} \in \bar{\Delta}$

$d_M(x,y) \le d_{\mathbb{E}^2}(\bar{x},\bar{y})$

(1)



$\Delta = \Delta(p,q,r)$ $\qquad$ $\bar{\Delta} = \bar{\Delta}(\bar{p},\bar{q},\bar{r})$



q_0, x_0, r_0 are not nec. collinear

Consider r_0' collinear with q_0, x_0 s.t. $d(x_0,r_0) = d(x_0,r_0')$

We claim that $d(p_0,r_0') \le d(p_0,r_0)$

In fact by construction $\gamma_0' \ge \gamma_0''$ $\overset{\text{Cor.}}{\Longrightarrow}$

$\Rightarrow d(p_0,r_0') \le d(p_0,r_0) = d(\bar{p},\bar{r})$

look at $\bar{\Delta}(\bar{p},\bar{q},\bar{r})$ &

$\Delta_{\mathbb{E}^2}(p_0,q_0,r_0')$

Want to see that $\bar{\beta} \ge \beta_0$

In fact $d(\bar{p},\bar{q}) = d(p_0,q_0)$

& $d(q_0,r_0') = d(\bar{q},\bar{r})$

Since $d(\bar{p},\bar{r}) \ge d(p_0,r_0')$

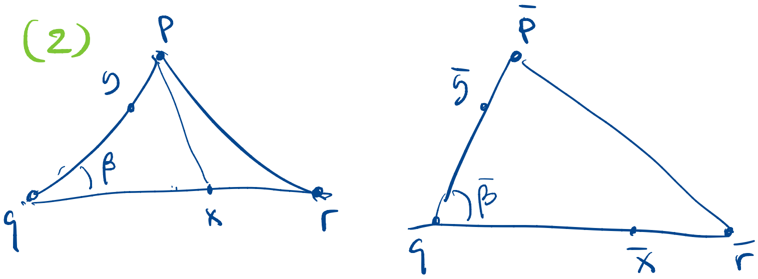
$\Rightarrow \bar{\beta} \ge \beta_0 \Rightarrow$

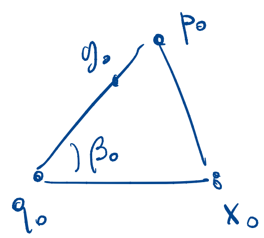
$d(\bar{p},\bar{x}) \ge d(p_0,x_0) \underset{\uparrow}{=} d(p,x)$

by construction

because

(2)





(1) $\Rightarrow$ $\overset{d(p_0,x_0)}{d(p,x)} \le$

$\le d(\bar{p},\bar{x})$ $\overset{\text{Cor}}{\Rightarrow}$

$\Rightarrow \beta_0 \le \bar{\beta}$ $\overset{\text{Cor}}{\Rightarrow}$

$\Delta_{\mathbb{E}^2}(\bar{q},\bar{x},\bar{y})$ &

$\Delta_{\mathbb{E}^2}(q_0,x_0,y_0) \Rightarrow$

$\Rightarrow d(x_0,y_0) \underset{\text{Cor}}{\le} d(\bar{x},\bar{y})$

(1) $\Vert$

$d(x,y)$ $\qquad\blacksquare$

Thm III.5 $\quad$ M RSS of non-cpt type, $o \in M$, $K = \mathrm{Stab}_G(M)$. Then

(1) Any cpt subgp $U < G$ has a fixed pt in M.

(2) $\{\mathrm{Stab}_G(p) : p \in M\} = \{\text{max. cpt. sbgps in } G\}$.

Pf (1) U cpt $\Rightarrow S = U_* o$ is bdd $\Rightarrow \exists$ circumcenter. S U-inv. $\Rightarrow$ the circum. is fixed by U.

(2) By (1) $\Rightarrow$ Any cpt $U < G$ is contained in some $\mathrm{Stab}_G(p)$. Max. cpt. are conjugate $\Rightarrow$ fix two pts in M (by transitivity).

Also if $p \ne q \Rightarrow \mathrm{Stab}_G(p) \ne \mathrm{Stab}_G(q)$.

If not $\Rightarrow \exists\, Z \in \mathfrak{p}$, $Z \ne 0$ s.t. $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k})Z = 0 \Rightarrow$

$\Rightarrow$ This would be true in one of the irred. sym. spaces.

$\Rightarrow$ This would have dim 1, impossible since M of non-cpt type $\blacksquare$

If M is of non-cpt type

$\Rightarrow M = G/K$, $K < G$ max. cpt.

If M^* is of cpt. type

$\Rightarrow M^* = G^*/K$, $K < G$ max. cpt.

III.2 Flats and rank

Defn A k-flat subspace (or k-flat) F in a RSS M is a totally geod. submanifold isometric to $\mathbb{E}^k$ for some $k \in \mathbb{N}$, that is $\forall\, p \in M$ and any $X, Y \in T_pM$ orthonormal $\Rightarrow K_p(\mathrm{span}\{X,Y\}) = 0$.

Rk 1-flat = geodesic

If K for M is strictly negative $\Rightarrow$ the only flats are geod.

$\Rightarrow$ hyp. space $\mathbb{R}, \mathbb{C}, \mathbb{H}$
plane $\mathbb{O}$.

Thm III.6 (1) (G,K) RSP (cpt or non-cpt. type), $d_e\pi : \mathfrak{p} \to T_oM$

$\mathrm{Exp}_o \circ d_e\pi : \mathfrak{p} \longrightarrow M$.

Then $F \ni o$ is a flat subspace $\Longleftrightarrow F = (\mathrm{Exp}_o \circ d_e\pi)(\mathfrak{a})$, where $\mathfrak{a} \subset \mathfrak{p}$ is an Abelian subspace.

Pf $(\Rightarrow)$ F totally geod $\Rightarrow$

$\Rightarrow F = (\mathrm{Exp}_o \circ d_e\pi)(\mathfrak{a})$, $\mathfrak{a} \subset \mathfrak{p}$ is a Lie triple system $\Rightarrow$

$\Rightarrow F$ is a RSS with RSP (G', K'), where $\mathfrak{g}' = \mathfrak{a} \oplus [\mathfrak{a},\mathfrak{a}]$

$K' = K \cap G'$. $\quad \forall\, X, Y \in \mathfrak{p}$

$K_o(\mathrm{span}\{X,Y\}) = -B_{\mathfrak{g}}([X,Y],[X,Y])$

So $K_o(\mathrm{span}\{X,Y\}) = 0 \Longleftrightarrow [X,Y] = 0$
$\forall\, X, Y \in \mathfrak{a}$ $\qquad \forall X, Y \in \mathfrak{a}$.

$\blacksquare$

Thm III.6 (2) If M is of non-cpt type $\Rightarrow \mathrm{Exp}_o \circ d_e\pi : \mathfrak{a} \to F$ is an isometry.

Pf $\forall\, X \in \mathfrak{p}$

$$d_X(\mathrm{Exp}_o \circ d_e\pi) = \underbrace{(d_o L_{\exp X} \circ d_e\pi)}_{\text{preserve the inner pr.}} \circ \underbrace{\sum_{n=0}^{\infty} \frac{(T_X|_{\mathfrak{p}})^n}{(2n+1)!}}_{\text{this is in general distance increas.}}$$

But if $X \in \mathfrak{a} \Rightarrow T_X|_{\mathfrak{a}} = 0$

$\Rightarrow \mathrm{Exp}_o \circ d_e\pi$ is a Riem. isom.

Since diffeo $\Rightarrow$ isometry.

$X \in \mathfrak{g}$

$\mathrm{Centr}_{\mathfrak{g}}(X) := \{Y \in \mathfrak{g} : [X,Y] = 0\}$

If $\mathfrak{a} \subset \mathfrak{p}$ Abelian and

$X \in \mathfrak{a} \Rightarrow \mathfrak{a} \subset \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$

Defn. $X \in \mathfrak{p}$ is a regular element if $\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is Abelian. (if not singular)

Thm III.7 M RSS of cpt or non-cpt. type, $\mathfrak{a} \subset \mathfrak{p}$ max. Abelian $\Rightarrow \exists\, X \in \mathfrak{p}$ s.t. $\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p} = \mathfrak{a}$