lecture

5 May 2021

From last week: A k-flat F in a RSS M is a tot. geod. subspace isom. to $\mathbb{E}^k$. Also $F \subset M$, $o \in F$ flat iff $F = (\mathrm{Exp}_o \circ d\pi)(\mathfrak{a})$, $\mathfrak{a} \subset \mathfrak{p}$ Abelian.

Claim Given RSS M the dim. of a max. Abelian subspace is well-defn.

Let $(\mathfrak{g}, \Theta)$ OSLA, $X \in \mathfrak{g}$.

$\mathrm{Centr}_{\mathfrak{g}}(X) = \{ Y \in \mathfrak{g} : [X, Y] = 0 \}$.

If $\mathfrak{a} \subset \mathfrak{p}$ Abelian & $X \in \mathfrak{a} \Rightarrow$

$\Rightarrow \mathfrak{a} \subset \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$.

Defn $X \in \mathfrak{p}$ is a regular element if $\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is Abelian.

A vector not regular is called singular.

Thm III.7 M RSS of cpt or non-cpt type, $G = \mathrm{Iso}(M)^\circ$, let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subspace $\Rightarrow \exists X \in \mathfrak{p}$ s.t.

$\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p} = \mathfrak{a}$.

Pf M RSS cpt type, $\mathfrak{a} \subset \mathfrak{p}$ max. Abelian. Let $\exp(\mathfrak{a}) < G$ be the corresp. connected Lie group. Let $A = \overline{\exp(\mathfrak{a})}$. To show: $\exp(\mathfrak{a})$ is closed hence a torus.

$\mathfrak{a} \subset \mathfrak{p} \Rightarrow \Theta(X) = -X \quad \forall X \in \mathfrak{a}$.

Since $\Theta = d_e\sigma \Rightarrow \sigma(\exp X) = (\exp X)^{-1}$

$\forall X \in \mathfrak{p} \Rightarrow \sigma(a) = a^{-1} \quad \forall a \in A$

$\Rightarrow A = \exp(\mathrm{Lie}\, A)$ with $\mathrm{Lie}\, A \subset \mathfrak{p}$ and Abelian $\Rightarrow \mathrm{Lie}\, A = \mathfrak{a} \Rightarrow$

$\Rightarrow A = \exp \mathfrak{a}$ closed $\Rightarrow \exp \mathfrak{a}$ cpt.

Since an Abelian Lie gp $\mathbb{T}^k \times \mathbb{R}^\ell$

$\Rightarrow A = \exp \mathfrak{a}$ is a torus $\Rightarrow$

$\Rightarrow \exists$ dense flow, that is $X \in \mathfrak{a}$ s.t. $\{\exp tX : t \in \mathbb{R}\}$ is dense in A.

$\Rightarrow$ If $X \in \mathfrak{a} \subset \mathfrak{p} \Rightarrow \mathfrak{a} \subset \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$.

To show the reverse inclusion, let $Y \in \mathrm{Centr}_{\mathfrak{g}}(X) \Rightarrow \{\exp sY : s \in \mathbb{R}\}$ commutes with $\{\exp tX : t \in \mathbb{R}\}$ hence with $A = \exp \mathfrak{a}$ by density

$\Rightarrow [Y, \mathfrak{a}] = 0$. Since $\mathfrak{a}$ is Ab.

$\Rightarrow \mathfrak{a} + Y\mathbb{R}$ is Abelian; since $\mathfrak{a}$ is max. Ab. and $\mathfrak{a} + Y\mathbb{R} \supseteq \mathfrak{a} \Rightarrow$

$\Rightarrow Y \in \mathfrak{a}$.

Let now M be of non-cpt type $(\mathfrak{g}, \Theta)$ the associated OSLA, M^* cpt dual with $(\mathfrak{g}^*, \Theta^*)$.

$\mathfrak{a} \subset \mathfrak{p}$ Abelian $\Leftrightarrow i\mathfrak{a} \subset i\mathfrak{p}$ Abelian

$\Rightarrow \exists\, iX \in i\mathfrak{p}$ s.t. $i\mathfrak{a} = \mathrm{Centr}_{\mathfrak{g}}(iX) \cap i\mathfrak{p}$

$\Rightarrow \mathfrak{a} = \mathrm{Centr}_{\mathfrak{g}}(iX) \cap \mathfrak{p}$). ■

Thm III.8 M RSS of non-cpt type, $\mathfrak{a}, \mathfrak{a}' \subset \mathfrak{p}$ two max. Abelian subalgebras. $\Rightarrow \exists k \in K$ s.t. $\mathfrak{a}' = \mathrm{Ad}_G(k)\mathfrak{a}$.

Defn. Let M be a RSS of cpt or non-cpt type. The rank $\mathrm{rk}(M)$ is the dimension of a max. flat in M.

Pf $X \in \mathfrak{a}$, $X' \in \mathfrak{a}'$ regular elem.

Consider $f : K \to \mathbb{R}$,

$f(k) := B_{\mathfrak{g}}(\mathrm{Ad}_G(k) X, X')$.

K cpt, f smooth $\Rightarrow f$ has a critical pt $k_0 \in K$, that is $\forall Z \in \mathfrak{k}$

$$\frac{d}{dt}\Big|_{t=0} B_{\mathfrak{g}}(\mathrm{Ad}_G(k_0 \exp tZ) X, X') = 0$$

$$0 = \frac{d}{dt}\Big|_{t=0} B_{\mathfrak{g}}(\mathrm{Ad}_G(k_0 \exp tZ) X, X')$$

$$= \frac{d}{dt}\Big|_{t=0} B_{\mathfrak{g}}(\mathrm{Ad}_G(k_0)\, \mathrm{Ad}_G(\exp tZ) X, X')$$

$$= B_{\mathfrak{g}}(\mathrm{Ad}_G(k_0)\, \mathrm{ad}_{\mathfrak{g}}(Z) X, X') =$$

$= B_{\mathfrak g}(\mathrm{Ad}_G(k_0)[Z,X], X')$

$= B_{\mathfrak g}([\mathrm{Ad}_G(k_0)(Z), \mathrm{Ad}_G(k_0)X], X')$

$= B_{\mathfrak g}(\underbrace{\mathrm{Ad}_G(k_0)(Z)}_{\in\, \mathfrak k}, \underbrace{[\mathrm{Ad}_G(k_0)(X), X']}_{\in\, \mathfrak k})$

$Z \in \mathfrak k \Rightarrow \mathrm{Ad}_G(k_0) Z \in \mathfrak k$

$X, X' \in \mathfrak p \Rightarrow \mathrm{Ad}_G(k_0)(X) \in \mathfrak p \Rightarrow [\mathfrak p, \mathfrak p] \subset \mathfrak k$.

But this was $\forall Z \in \mathfrak k$ & $B_{\mathfrak g}$ is non-degenerate $\Rightarrow [\mathrm{Ad}_G(k_0)(X), X'] = 0$

$\Rightarrow \mathrm{Ad}_G(k_0)(X) \in \mathrm{Centr}_{\mathfrak g}(X') \cap \mathfrak p = \mathfrak a'$

But $\mathfrak a'$ is Abelian $\Rightarrow$ every element in $\mathfrak a'$ commutes with $\mathrm{Ad}_G(k_0)(X)$

$\Rightarrow \mathfrak a' \subset \mathrm{Centr}_{\mathfrak g}(\mathrm{Ad}_G(k_0) X) \cap \mathfrak p =$

$= \mathrm{Ad}_G(k_0)(\mathrm{Centr}_{\mathfrak g}(X) \cap \mathfrak p) =$

$= \mathrm{Ad}_G(k_0)(\mathfrak a)$.

By maximality $\mathfrak a' = \mathrm{Ad}_G(k_0)(\mathfrak a)$. ∎

Corollary III.9 Every geodesic is contained in at least one flat. It is contained in exactly one flat iff the $X \in \mathfrak p$ that defines it is regular.

Pf Let $\gamma(t) := \exp(tX) \cdot o$.

$\Rightarrow \gamma \subset \exp \mathfrak a$ $\forall$ Abelian $\mathfrak a$ with $X \in \mathfrak a$.

($\Leftarrow$) X regular, $\mathfrak a, \mathfrak a' \subset \mathfrak p$ max. Abelian s.t. $\gamma \subset \exp \mathfrak a \cdot o$ and $\gamma \subset \exp \mathfrak a' \cdot o$. Since $X \in \mathfrak a'$ and $\mathfrak a'$ Abelian $\Rightarrow \mathfrak a' \subset \underbrace{\mathrm{Centr}_{\mathfrak g}(X) \cap \mathfrak p}_{\mathfrak a}$

$\Rightarrow \mathfrak a = \mathfrak a'$.

($\Rightarrow$) Assume $\gamma(t) = (\exp tX) \cdot o$ is contained in exactly one flat & X not regular, that is $\mathrm{Centr}_{\mathfrak g}(X) \cap \mathfrak p$ is not Abelian.

$\mathfrak a$ Abelian, $X \in \mathfrak a \Rightarrow \mathfrak a \subsetneq \mathrm{Centr}_{\mathfrak g}(X) \cap \mathfrak p$.

Let $X' \in \mathrm{Centr}_{\mathfrak g}(X) \cap \mathfrak p$ but $X' \notin \mathfrak a$.

Since $[X, X'] = 0 \Rightarrow \mathrm{span}\{X, X'\}$ is Abelian $\Rightarrow$ let $\mathfrak a'$ be a maximal Abelian that contains $\mathrm{span}\{X, X'\}$. But $X' \in \mathfrak a'$, but $X' \in \mathfrak a \Rightarrow \mathfrak a' = \mathfrak a$. But $X \in \mathfrak a' \Rightarrow \gamma \subset \exp \mathfrak a'$ ↯. ∎

Example (1) $M = SL(n,\mathbb R)/SO(n,\mathbb R)$

$\mathfrak p = \{X \in \mathfrak{sl}(n,\mathbb R) : {}^tX = X\}$.

We claim that

$\mathfrak a = \{\mathrm{diag}(t_1, \dots, t_n) : \sum t_j = 0, \ t_j \in \mathbb R\}$

is maximal Abelian.

$H \in \mathfrak a$ is regular iff $t_i \neq t_j$ if $i \neq j$.

In fact $[H, X]_{ij} = (t_j - t_i) X_{ij}$

$\Rightarrow \mathrm{Centr}_{\mathfrak g}(H) = \mathfrak a$.

$\Rightarrow \mathrm{rk}(\mathfrak{sl}(n,\mathbb R)) = n-1$.

Recall $M = \mathrm{Pos}_1(n)$ = positive matrices with det. 1 with the $SL(n,\mathbb R)$-action by conjugation $g \cdot p = g^t p \, g$. If $\mathrm{Id}_n \in \mathrm{Pos}_1(n)$ is the basept a max. flat is

$F = \exp \mathfrak a \cdot \mathrm{Id}_n =$

$= \{\mathrm{diag}(e^{2t_1}, \dots, e^{2t_n}) : \sum t_j = 0\}$

$= \{\mathrm{diag}(\lambda_1, \dots, \lambda_n) : \lambda_j > 0, \ \prod \lambda_j = 1\}$.

Now we show how if $X \in \mathfrak a$ has non-distinct e.values it is contained in a 1-par. family of flats. Set $n = 3$.

$X = \mathrm{diag}(1,1,2) \in \mathfrak{a}$.

$$k_\theta := \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K$$

$\Rightarrow \mathrm{Ad}_G(k_\theta) X = X$.

Let Y be a vector orthogonal to X (e.g. $Y = (2,0,1)$).

$\Rightarrow \mathfrak{a} = \mathrm{span}\{X, Y\}$ is a max. Abelian subalg. containing $\gamma = \exp tX \subset \exp \mathfrak{a}$.

Moreover $\mathrm{span}\{X, \mathrm{Ad}_G(k_\theta)Y\} \supset \gamma$.

(2) $\mathrm{rk}\left(SO(p,q) / S(O(p) \times O(q)) \right) = \min\{p,q\}$

(3) $\mathrm{rk}\left(Sp(2q, \mathbb{R}) \right) = q$.

III.3 Roots and root spaces

$(\mathfrak{g}, \Theta)$ OSLA of non-cpt type; $\mathfrak{a} \subset \mathfrak{g}$ consider $\langle X, Y \rangle := -B_\mathfrak{g}(X, \Theta Y)$.

Lemma III.10 If $X \in \mathfrak{p}$ the endom. $\mathrm{ad}_\mathfrak{g}(X) \in \mathrm{End}(\mathfrak{g})$ is self-adj. w.r.t. $\langle , \rangle$.

Pf Verification ∎

If $\mathfrak{a} \subset \mathfrak{p}$ is Abelian $\Rightarrow$
$\Rightarrow \{\mathrm{ad}_\mathfrak{g}(X) : X \in \mathfrak{a}\}$ is a comm. family of diag. endom. $\Rightarrow$ they are simoultaneously diagon.

Defn. $\alpha \in \mathfrak{a}^*$,

$$\mathfrak{g}_\alpha := \{ X \in \mathfrak{g} : \mathrm{ad}_\mathfrak{g}(H) X = \alpha(H) X \ \forall H \in \mathfrak{a} \}$$

Lemma III.11

(1) $\alpha, \beta \in \mathfrak{a}^* \Rightarrow [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

(2) $\Theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha} \quad \forall \alpha \in \mathfrak{a}^*$.

Pf (1) $H \in \mathfrak{a}$, $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$

$$\Rightarrow \mathrm{ad}_\mathfrak{g}(H)[X,Y] = [\mathrm{ad}_\mathfrak{g}(H)X, Y] + [X, \mathrm{ad}_\mathfrak{g}(H)(Y)] =$$
$$= \alpha(H)[X,Y] + \beta(H)[X,Y] = (\alpha+\beta)(H)[X,Y].$$

(2) $H \in \mathfrak{a} \subset \mathfrak{p} \Rightarrow \Theta(H) = -H$.

If $X \in \mathfrak{g}_\alpha \Rightarrow$

$$\Rightarrow \mathrm{ad}_\mathfrak{g}(H)\Theta(X) = [H, \Theta(X)] = [\Theta(H), X] = -[H,X] = -\mathrm{ad}_\mathfrak{g}(H) X.$$ ∎

Defn. A root of $\mathfrak{a}$ in $\mathfrak{g}$ is a non-zero linear form on $\mathfrak{a}^*$ s.t. $\mathfrak{g}_\alpha \neq (0)$. Then $\mathfrak{g}_\alpha$ is the associated root space.

If $\Sigma \subset \mathfrak{a}^* \setminus \{0\}$ is the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$ $\Rightarrow$

$$\Rightarrow \mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma}^{\oplus} \mathfrak{g}_\alpha$$

and the decomposition is orthogonal w.r.t. $\langle , \rangle$.

Let $\mathfrak{a}^* \longrightarrow \mathfrak{a}$, $\alpha \longmapsto H_\alpha$ be the isom.

$$\alpha(H) = B_\mathfrak{g}(H, H_\alpha)$$

Thm III.12 Let $\mathfrak{a} \subset \mathfrak{p}$ be a max. Abelian subspace, $\Sigma \subset \mathfrak{a}^*$ the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$. Then Σ is a root system, that is

(1) Σ spans $\mathfrak{a}^*$

(2) For every $\alpha, \beta \in \Sigma$,
$$\beta - \frac{2 B_\mathfrak{g}(H_\alpha, H_\beta)}{B_\mathfrak{g}(H_\alpha, H_\alpha)} \alpha \in \Sigma$$

(3) $\dfrac{2\, B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} \in \mathbb{Z}$.

Rem $\mathfrak{a}^* \longrightarrow \mathfrak{a}$

$\alpha \longmapsto H_\alpha$

Let $\langle\, , \rangle$ be the inner product on $\mathfrak{a}^*$, $\langle \alpha, \beta \rangle := B_{\mathfrak{g}}(H_\alpha, H_\beta)$

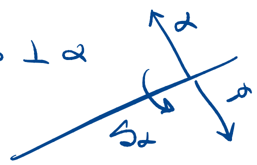
$\Rightarrow$ $\forall \alpha \in \mathfrak{a}^*$, $S_\alpha : \mathfrak{a}^* \longrightarrow \mathfrak{a}^*$

$$S_\alpha(\beta) = \beta - \frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha \rangle} \cdot \alpha$$

is the reflection on the hyperplane orthogonal to α.

$S_\alpha(\alpha) = -\alpha$

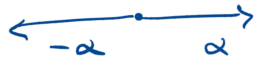
$S_\alpha(\beta) = \beta$ if $\beta \perp \alpha$



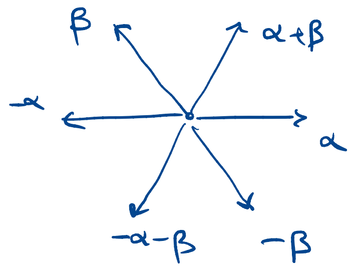
Exercise

Verify that the root system of

(1) $\mathfrak{sl}(2,\mathbb{R})$



(2) $\mathfrak{sl}(3,\mathbb{R})$



From now on $\mathfrak{a}$ = max. Abel. subalg.

Lemma III.13 $\mathfrak{a}$ max. Abelian $\Rightarrow$

$\Rightarrow$ $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$.

Pf $\mathfrak{a} \subset \mathfrak{g}_0 \cap \mathfrak{p} = \{ X \in \mathfrak{p} : [X, H] = 0 \ \forall H \in \mathfrak{a} \}$

If on the other hand $X \in \mathfrak{g}_0 \cap \mathfrak{p}$

$\Rightarrow$ $\mathfrak{a} + X$ Abelian $\overset{\text{max}}{\Rightarrow}$ $X \in \mathfrak{a}$. $\blacksquare$

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

$\alpha \in \Sigma \Rightarrow -\alpha \in \Sigma$.

Strategy:

Find $X_\alpha \in \mathfrak{g}_\alpha$, $X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ and $H_\alpha \in \mathfrak{a}$ s.t.

$\{X_\alpha, X_{-\alpha}, H_\alpha\}$ span a copy of $\mathfrak{sl}(2,\mathbb{R})$