Exercise Class Symmetric Spaces

6.5.21

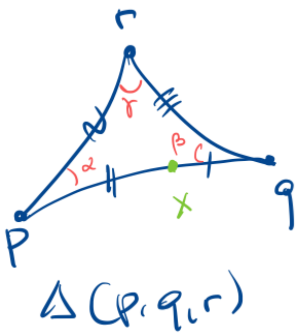
Recap:

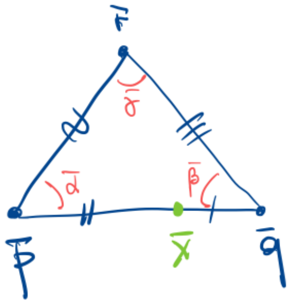
(A) CAT(0) Spaces:

Defn: A geod. m. sp. is CAT(0) if $\forall$ $\Delta(p,q,r)$ with comparison triangle $\bar\Delta(\bar p,\bar q,\bar r)$ and pts $x,y \in \Delta(p,q,r)$ with corresp. pts $\bar x,\bar y \in \bar\Delta$ the inequality

$$d_X(x,y) \le d_{\mathbb{E}^2}(\bar x,\bar y)$$

In a CAT(0) space triangles are thin, that is $\alpha \le \bar\alpha$, $\beta \le \bar\beta$, $\gamma \le \bar\gamma$.





Slogan: Geodesic triangles are thinner than their comparison triangle in $\mathbb{E}^2$.

Rmk: ∘ In fact, the above definition of CAT(0) spaces extends to CAT(k) spaces for $k \in \mathbb{R}$.

Instead of comparing geodesic triangles to comparison triangles in $\mathbb{E}^2$ we compare them to comparison triangles in $\mathbb{S}^2_k$ (if $k>0$) or $\mathbb{H}^2_k$ (if $k<0$).

$\uparrow$ (rescaled) sphere with sect. curv. $\equiv k$

$\uparrow$ (rescaled) hyperbolic space with sect. curv $\equiv k$

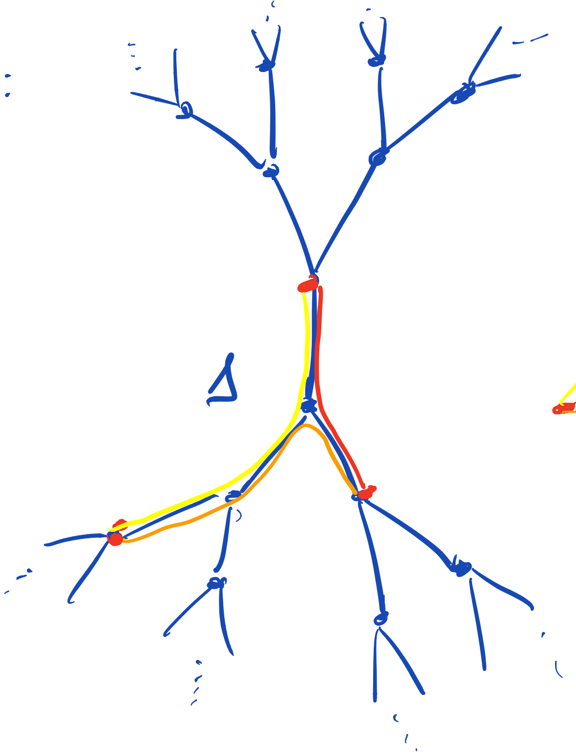
∘ This is a key notion in modern metric geometry!

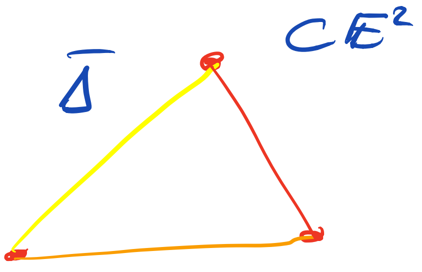
Examples:

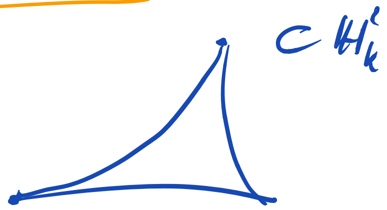
- $\mathbb{E}^n$ is CAT(0)
- $\mathbb{H}^n$ is CAT(0) (and even CAT(-1) by defn)
- Any symmetric space of non-cpt type is CAT(0) (Thm III.3)
- More generally, any complete simply connected Riem. mfld with sectional curvature ≤ 0 is CAT(0)!

∘ "Non-smooth example":

Trees:







Indeed, any $\Delta \subset T$ is thinner than the comp. $\bar{\Delta} \subset \mathbb{E}^2$.

Actually, trees are much more neg. curved:

Every tree is CAT($-k$) $\forall\, k>0$!

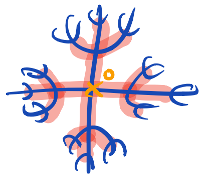
⌈In fact, (regular locally finite) trees are in many ways⌉ similar to symmetric spaces:

∘ $G := \mathrm{Aut}(T) \curvearrowright T$ transitively
$\uparrow$ graph automorphisms.

∘ $G_v \leq G$ stabilizer subgroups of vertices $v \in V(T)$ are maximal compact subgroups.

Yet, they are very different from a topological perspective:

A nbhd. system of the identity $id \in Aut(T)$ is given by the compact subgroups:



$$N_r := \{\varphi \in Aut(T) : \varphi(v) = v \quad \forall v \in B_r(o) \subset T\}.$$

$\forall r > 0.$

In particular, every identity neighborhood contains a subgroup! This is in stark contrast to Lie groups which do NOT admit small subgroups. (seen in "Intro. to Lie gps." last semester).

Rmk: Automorphism groups of trees are important examples of totally disconnected locally compact (tdlc) groups and are the subject of ongoing research.

(B) Flats and Rank:

Defn A k-flat subspace (or k-flat) F in a RSS M is a totally geod. submanifold isometric to $\mathbb{E}^k$ for some $k \in \mathbb{N}$, that is $\forall$ $p \in M$ and any $X, Y \in T_pM$ orthonormal $\Rightarrow$ $K_p(\operatorname{span}\{X, Y\}) = 0$.

Thm III.6 (1) (G, K) RSP (cpt or non-cpt. type), $d_e\pi : \mathfrak{p} \to T_oM$, $\mathrm{Exp}_o \circ d_e\pi : \mathfrak{p} \longrightarrow M$.
Then $F \ni o$ is a flat subspace $\Leftrightarrow$ $F = (\mathrm{Exp}_o \circ d_e\pi)\,\mathfrak{a}$, where $\mathfrak{a} \subset \mathfrak{p}$ is an Abelian subspace.

$$\begin{tikzcd}
\mathfrak{g} & \\
\mathfrak{p} \arrow[u, phantom, sloped, "\subset"] \arrow[r, "d\pi"] \arrow[rdd, bend left] & T_oM \arrow[dd, "\mathrm{Exp}_o"] \\
\mathfrak{a} \arrow[u, phantom, sloped, "\subset"] & \\
G \arrow[r] & G/K \cong M \\
& F \arrow[u, phantom, sloped, "\subset"]
\end{tikzcd}$$

Defn $X \in \mathfrak{p}$ is a regular element if $\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is Abelian.
A vector not regular is called singular.

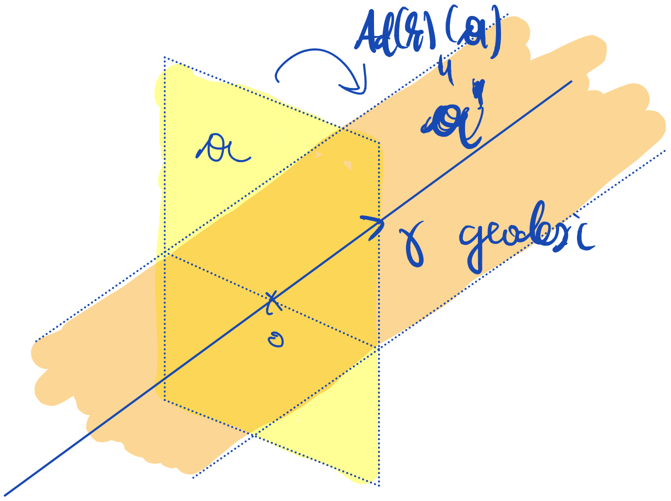
$\mathrm{Centr}_{\mathfrak{g}}(X) = \{Y \in \mathfrak{g} \mid [X, Y] = 0\}$

(If $X \in \underset{\text{abelian}}{\mathfrak{a}} \subset \mathfrak{p}$, then $\mathfrak{a} \subseteq \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$)

Thm III.8 M RSS of non-cpt type, $\mathfrak{a}, \mathfrak{a}' \subset \mathfrak{p}$ two max. Abelian subalgebras. $\Rightarrow$

$\exists\, k \in K$ s.t. $\mathfrak{a}' = \mathrm{Ad}_G(k)\,\mathfrak{a}$.

Corollary III.9 Every geodesic is contained in at least one flat. It is contained in exactly one flat iff the $X \in \mathfrak{p}$ that defines it is regular.



Defn. Let M be a RSS of cpt or non-cpt type. The rank $\mathrm{rk}(M)$ is the dimension of a max. flat in M.

— BREAK —

Exercise Sheet 4:

Exercise 1.(Complexification and Killing form):

Let $\mathfrak{l}_0$ be a Lie algebra over $\mathbb{R}$ and let $\mathfrak{l}$ be the complexification of $\mathfrak{l}_0$. Let K_0, K and $K^{\mathbb{R}}$ denote the Killing forms of the Lie algebras $\mathfrak{l}, \mathfrak{l}_0$ and $\mathfrak{l}^{\mathbb{R}}$, respectively. Show that:

a) $K_0(X,Y) = K(X,Y)$ for all $X, Y \in \mathfrak{l}_0$;

b) $K^{\mathbb{R}}(X,Y) = 2 \cdot \mathrm{Re}(K(X,Y))$ for all $X, Y \in \mathfrak{l}^{\mathbb{R}}$.

Sol: (a): ✓

(b): $\mathcal{B} = \{X_1, \dots, X_n\}$ $\mathbb{C}$-basis $\mathfrak{l}$

Write $$ad(X)\, ad(Y)\, X_i = \sum_{j=1}^{n} \underbrace{\alpha_{ij}}_{=\beta_{ij} + i \cdot \gamma_{ij}} \cdot X_j \qquad \circledast$$

$$\Rightarrow M_{\mathcal{B}}(ad(X)ad(Y)) = \underbrace{A}_{=(\alpha_{ij})} = B + i \cdot C$$

$$ad(X)\, ad(Y)(i \cdot X_i) = \sum -\gamma_{ij} \cdot X_j + i \sum \beta_{ij} \cdot X_j \qquad \circledast$$

$\mathcal{C} := \{X_1, \dots, X_n, i \cdot X_1, \dots, i \cdot X_n\}$ $\mathbb{R}$-basis for $\mathfrak{l}^{\mathbb{R}}$.

$$\circledast : M_{\mathcal{C}}(ad(X)ad(Y)) = \begin{pmatrix} B & -C \\ C & B \end{pmatrix}.$$

$$\Rightarrow K^{\mathbb{R}}(X,Y) = 2\, tr(B) = 2 \cdot Re\, tr(A) = 2\, Re\, K(X,Y).$$

$\square$

Exercise 2.(Semisimple OSLAs):

Let $(\mathfrak{l},\Theta)$ be an orthogonal symmetric Lie algebra with $\mathfrak{l}$ semisimple. Show that:

a) $\mathfrak{u}$ equals its normalizer in $\mathfrak{l}$;

b) if $\mathfrak{u}$ contains no ideal in $\mathfrak{l}$ then $[\mathfrak{e},\mathfrak{e}]=\mathfrak{u}$.

Sol: (a) $N_\mathfrak{l}(\mathfrak{u}) = \{X \in \mathfrak{l} \mid [X,\mathfrak{u}] \subset \mathfrak{u}\}$.

WTS: $\mathfrak{u} = N_\mathfrak{l}(\mathfrak{u})$. $\mathfrak{u} \subseteq N_\mathfrak{l}(\mathfrak{u})$ because it's subalgebra.

"$\supseteq$": Let $X = X_\mathfrak{u} + X_\mathfrak{e} \in N_\mathfrak{l}(\mathfrak{u})$

$\Longrightarrow$ $X_\mathfrak{e} \in N_\mathfrak{l}(\mathfrak{u})$, indeed:

$$[X_\mathfrak{e}, Y] = [X - X_\mathfrak{u}, Y] = \underbrace{[X,Y]}_{\in \mathfrak{u}} - \underbrace{[X_\mathfrak{u},Y]}_{\in \mathfrak{u}} \in \mathfrak{u} \quad \forall Y \in \mathfrak{u}.$$

WLOG: $X \in \mathfrak{e}$.

$$\underbrace{[X,\mathfrak{u}]}_{\subset [\mathfrak{e},\mathfrak{u}] \subset \mathfrak{e}} \subset \mathfrak{u} \Longrightarrow [X,\mathfrak{u}] \subset \mathfrak{u} \cap \mathfrak{e} = \{0\}.$$

0 bec. $\mathfrak{l}$ is semisimple

Further decompose: $X = \overset{0}{\cancel{X_0}} + X_+ + X_- \in \mathfrak{e}_0 \oplus \mathfrak{e}_+ \oplus \mathfrak{e}_-$

$$0 = \underbrace{[X_+,\mathfrak{u}]}_{\in \mathfrak{e}_+} + \underbrace{[X_-,\mathfrak{u}]}_{\in \mathfrak{e}_-}$$

$\Longrightarrow$ $\mathrm{ad}(X_+)^2\mathfrak{u} = 0$, $\mathrm{ad}(X_-)^2\mathfrak{u} = 0$

$$\mathrm{ad}(X_+)^2\mathfrak{e} = [X_+, \underbrace{[X_+,\mathfrak{e}]}] \in [X_+,\mathfrak{u}] = 0$$

$\subset u$

$$\rightsquigarrow \mathrm{ad}(X_-)^2 e = 0$$

$$B(X_+, X_+) = \mathrm{tr}\, \mathrm{ad}(X_+)^2 = 0 \quad , \quad B|_{e^+ \times e^+} >> 0$$

$$B(X_-, X_-) = \mathrm{tr}\, \mathrm{ad}(X_-)^2 = 0 \quad , \quad B|_{e^- \times e^-} << 0$$

$$\Rightarrow \quad X_+ = 0 = X_- \quad \Rightarrow \quad X = 0.$$

□

(b) Decompose: $e = \overset{0}{\cancel{e_0}} + e_+ + e_- = e_- + e_+$

$$u = u_0 + u_+ + u_-$$

$$[u_0, e] = [u_0, e_+] + [u_0, e_-] \in e_0 = \{0\}.$$

$\Rightarrow \quad u_0 \trianglelefteq \ell$ ideal

Finally: $[e,e] = [e_+, e_+] + [e_+, e_-] + [e_-, e_-]$

$$= u_+ + u_- = u.$$

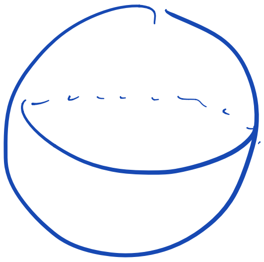
□

Exercise 3.($\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})^{\mathbb{R}}$):

Exhibit an explicit isomorphism between $\mathfrak{so}(1,3)$ and $\mathfrak{sl}(2,\mathbb{C})$.

Hint: Consider the the vector space V of 2×2-skew-Hermitian matrices and endow it with the quadratic form $q(v) := \det(v)$. Now, let $\mathrm{SL}(2,\mathbb{C})$ act on V via $g.v := gv\bar{g}^t$.

Aside: $\mathbb{H}^3 = SO^0(1,3)/SO(3)$ there's a ball model for $\mathbb{H}^3$



We can identify $\partial\mathbb{H}^3 \cong S^2 \cong \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

$SL_2\mathbb{R} \curvearrowright \mathbb{H} \subset \mathbb{C}$ via Möbius trafo.

$SL_2\mathbb{C} \curvearrowright \hat{\mathbb{C}} \cong \partial\mathbb{H}^3$ via — " —

$\uparrow$ induced action of $\mathrm{Isom}^+(\mathbb{H}^3) \cong SO^0(1,3)$ on $\partial\mathbb{H}^3$.

Sol: Write:

$$v = \begin{pmatrix} i(x_1 - x_3) & -x_2 + i\cdot x_4 \\ x_2 + i\cdot x_4 & i(x_1 + x_3) \end{pmatrix} \in V = \{v \mid v = \bar{v}^t\}$$

$$(x_1, \ldots, x_4 \in \mathbb{R})$$

$$q(v) = \det(v) = -x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

Check: $SL_2(\mathbb{C}) \curvearrowright V$:

$$(\overline{g.v})^t = (\overline{g v \bar{g}^t})^t = g\bar{v}^t\bar{g}^t = g v \bar{g}^t. \quad ✓$$

$$q(g.v) = \det(g v \bar{g}^t) = \underbrace{\det(g)}_{=1} \det(v) \underbrace{\det(\bar{g}^t)}_{=1}$$

$$= \det(v) = q(v).$$

$\Longrightarrow \quad \varphi : SL_2(\mathbb{C}) \longrightarrow SO(1,3), \quad \varphi(g)(v) = g.v.$

Direct computation: $\ker\varphi = \{\pm I\}$. discrete.

$\Longrightarrow \quad d\varphi : \mathfrak{sl}_2(\mathbb{C}) \longrightarrow \mathfrak{so}(1,3)$ injective.

$\dim \mathfrak{sl}_2\mathbb{C} = \dim \mathfrak{so}(1,3) \Longrightarrow d\varphi$ isom. $\square$

Exercise 4.(Duality of $\mathbb{S}^n$ and $\mathbb{H}^n$):

Show that the symmetric spaces $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $\mathbb{H}^n \cong \mathrm{SO}(1,n)^\circ/\mathrm{SO}(n)$ are dual to each other.

Sol: Done in the lecture. $\square$