Lecture

12 Mai 2021

Last week $(\mathfrak{g}, \Theta)$ OSLA non-cpt type

$\langle X, Y\rangle := -B_{\mathfrak{g}}(X, \Theta(Y)) \quad \forall X, Y \in \mathfrak{g}$

If $\mathfrak{a} \subset \mathfrak{p}$ max. Abelian, $\alpha \in \mathfrak{a}^* \Rightarrow$

root space associated to a root α of $\mathfrak{a}$ in $\mathfrak{g}$

$\mathfrak{g}_\alpha := \{X \in \mathfrak{g} : \mathrm{ad}_{\mathfrak{g}}(H)X = \alpha(H)X \quad \forall H \in \mathfrak{a}\}$

$$\Rightarrow \mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha \quad \text{orthog. w.r.t. } \langle\ ,\ \rangle$$

$\Sigma \subseteq \mathfrak{a}^* \setminus \{0\}$

Three reasons to study root spaces/systems:

- Classif. of complex s.s. Lie alg.
- Study f.d. repr. of s.s. gps.
- Study geom. properties of RSS

Recall $X \in \mathfrak{p}$ regular if $\mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$ Abelian. If $X \in \mathfrak{a} \Rightarrow \mathfrak{a} \subset \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$

$\Rightarrow \mathfrak{a}$ max. ab. $\Leftrightarrow \exists X \in \mathfrak{a}$ s.t. $\mathfrak{a} = \mathrm{Centr}_{\mathfrak{g}}(X) \cap \mathfrak{p}$.

Lemma III.14 A vector $H \in \mathfrak{a} \setminus \{0\}$ is regular iff $\alpha(H) \neq 0 \ \forall \alpha \in \Sigma$

Pf ($\Rightarrow$) $H \in \mathfrak{a}$ regular $\Rightarrow \mathfrak{a} = \mathrm{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p}$ and suppose $\exists \alpha \in \Sigma$ s.t. $\alpha(H) = 0$. Let $X \in \mathfrak{g}_\alpha$:

$X = X_{\mathfrak{k}} + X_{\mathfrak{p}}, \quad X_{\mathfrak{k}} = X + \Theta(X), \ X_{\mathfrak{p}} = X - \Theta(X)$

$\Rightarrow \forall A \in \mathfrak{a} \quad \mathrm{ad}_{\mathfrak{g}}(A)(X) = \alpha(A)X$

$\mathrm{ad}_{\mathfrak{g}}(\underset{\mathfrak{a} \subset \mathfrak{p}}{A})(\underset{\mathfrak{k}}{X_{\mathfrak{k}}} + \underset{\mathfrak{p}}{X_{\mathfrak{p}}}) = \alpha(A)(X_{\mathfrak{k}} + X_{\mathfrak{p}})$ Since $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ $\Rightarrow$

$\Rightarrow \mathrm{ad}(A)X_{\mathfrak{p}} = \alpha(A)X_{\mathfrak{k}} \quad \forall A \in \mathfrak{a}$

If $A = H$, since $\alpha(H) = 0 \Rightarrow$

$\Rightarrow \mathrm{ad}(H)X_{\mathfrak{p}} = 0 \Rightarrow X_{\mathfrak{p}} \in \mathrm{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p} = \mathfrak{a}$

$\Rightarrow 0 = \mathrm{ad}(A)X_{\mathfrak{p}} = \alpha(A)X_{\mathfrak{k}} \quad \forall A \in \mathfrak{a}$

But $\alpha \in \Sigma$, that is $\alpha \not\equiv 0 \Rightarrow X_{\mathfrak{k}} = 0$

$\Rightarrow X = X_{\mathfrak{p}} \in \mathfrak{a} \Rightarrow \mathfrak{g}_\alpha \subset \mathfrak{a} \underset{\text{III.13}}{\subset} \mathfrak{g}_0$ ↯

($\Leftarrow$) Assume $\alpha(H) \neq 0 \ \forall \alpha \in \Sigma$ and that $\mathrm{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p}$ not Abelian.

But $H \subset \mathfrak{a} \subsetneq \mathrm{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p}$.

Let $Y \in \mathrm{Centr}_{\mathfrak{g}}(H) \cap \mathfrak{p}$, $Y \notin \mathfrak{a}$.

$Y = Y_0 + \sum_{\alpha \in \Sigma} Y_\alpha, \quad Y_\alpha = \text{proj. of } Y \text{ on } \mathfrak{g}_\alpha$

$[H, Y_\alpha] = \alpha(H)Y_\alpha$

$[H, Y_0] = 0 \Rightarrow Y_0 \in \mathrm{Centr}_{\mathfrak{g}}(H)$

Since $Y \in \mathrm{Centr}_{\mathfrak{g}}(H) \Rightarrow$

$$\Rightarrow 0 = [H, Y] = \underbrace{[H, Y_0]}_{=0} + \sum_{\alpha \in \Sigma}[H, Y_\alpha] =$$

$$= \sum_{\alpha \in \Sigma} \alpha(H) Y_\alpha \Rightarrow Y_\alpha = 0 \ \forall \alpha \in \Sigma$$

$\Rightarrow Y = Y_0 \in \mathfrak{a}$ ↯ ∎

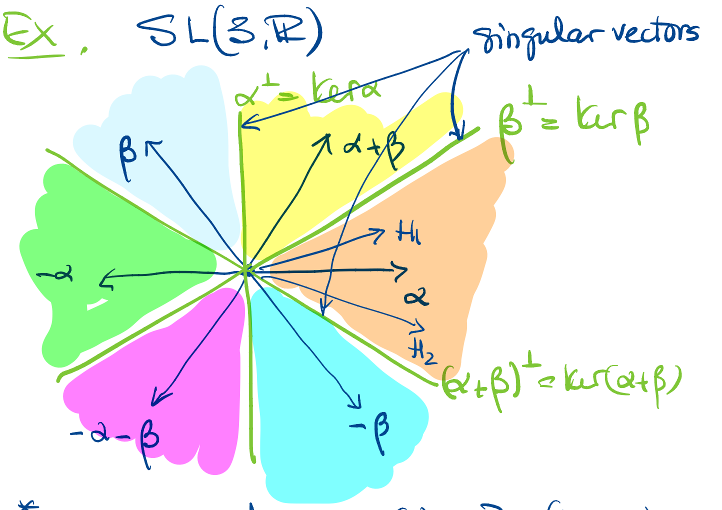
Corollary III.15 $\mathfrak{a}_{reg} = \mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} \ker \alpha$

set of regular elements in $\mathfrak{a}$.

Defn. Let $\mathfrak{a} \subset \mathfrak{p}$ max. Abel. A connected component of $\mathfrak{a}_{reg}$ is called a Weyl chamber in $\mathfrak{a}$.

Fact A Weyl chamber can also be described as an equiv. class, where $H_1 \sim H_2 \Leftrightarrow \alpha(H_1)$ and $\alpha(H_2)$ have the same sign $\forall \alpha \in \Sigma$.

Ex. $SL(3, \mathbb{R})$



$\mathfrak{a}^* \to \mathfrak{a}$ where $\alpha(H) = B_{\mathfrak{g}}(H, H_\alpha)$

$\alpha \longmapsto H_\alpha$ $\quad H_\alpha$ is a root vector

Ex. $(\mathfrak{sl}(n,\mathbb{R}), \Theta)$, $\Theta(X) = -X^t$.

$E_{ij} :=$ elem. matrix $= \begin{cases} 1 & \text{in } (i,j) \\ 0 & \text{otherwise} \end{cases}$

$H_j := E_{jj} - E_{j+1,j+1} \Rightarrow \{E_{ij},\ i\neq j,\ H_j,\ j=1,\dots,n-1\}$ is a basis of $\mathfrak{sl}(n,\mathbb{R})$

Ex. $n=2$ $\quad e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\mathfrak{a}$ max. Ab. $\Rightarrow \mathfrak{a} = \{\mathrm{diag}(t_1, \dots, t_n) : \sum t_j = 0,\ t_j \in \mathbb{R}\}$

If $H \in \mathfrak{a}$, $H = \mathrm{diag}(t_1, \dots, t_n) \Rightarrow$

$$\Rightarrow \left.\begin{array}{l} \mathrm{ad}(H) E_{ij} = (t_i - t_j) E_{ij} \\ \mathrm{ad}(H) H_j = 0 \end{array}\right\} \forall\, i \neq j$$

$\Rightarrow \exists\ n(n-1)$ non-zero roots

$\alpha_{ij}(A) := A_{ii} - A_{jj}$

and $n(n-1)$ one-dim. root spaces,

$\mathfrak{g}_{ij} := \mathfrak{g}_{\alpha_{ij}} \Rightarrow \mathfrak{sl}(n,\mathbb{R}) = \mathfrak{a} \oplus \sum_{i\neq j} \mathbb{R}\,\mathfrak{g}_{ij}$

Claim $\exists$ 1-1 corr. between Weyl chambers and elements in S_n

$$\left.\begin{array}{ll} A = \mathrm{diag}(\lambda_1, \dots, \lambda_n) & \lambda_i \neq \lambda_j \\ B = \mathrm{diag}(\mu_1, \dots, \mu_n) & \mu_i \neq \mu_j \end{array}\right\} A, B \text{ reg.}$$

$\exists!\ \sigma \in S_n$ s.t. $\lambda_{\sigma(1)} > \dots > \lambda_{\sigma(n)}$

and $\tau \in S_n$ s.t. $\mu_{\tau(1)} > \dots > \mu_{\tau(n)}$

A, B are in the same Weyl chambers $\Leftrightarrow (\lambda_i - \lambda_j)(\mu_i - \mu_j) > 0$ $\forall\, i \neq j$.

One can show that $\sigma = \tau \Rightarrow$

$\Rightarrow \mathfrak{a}^+ := \{\mathrm{diag}(t_1, \dots, t_n) : t_1 > \dots > t_n\}$

Thm III.12 $\mathfrak{a} \subset \mathfrak{p}$ max. Ab. Then Σ is a root system, that is

(1) Σ spans $\mathfrak{a}^*$

(2) $\forall\, \alpha, \beta \in \Sigma \Rightarrow \beta - \frac{2 B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} \alpha \in \Sigma$

(3) $\frac{2 B_{\mathfrak{g}}(H_\alpha, H_\beta)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} \in \mathbb{Z}$.

Recall $\mathfrak{sl}(2,\mathbb{R}) = \mathrm{span}\left\{ \underset{e_+}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{e_-}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{h}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \right\}$

$[e_+, e_-] = h$

$[h, e_+] = 2e_+$

$[h, e_-] = -2e_-$

Lemma III.16 $(\mathfrak{g}, \Theta)$ OSLA of non-cpt type, $\mathfrak{a}$ max. Abelian. If $\alpha \in \Sigma$ and $X \in \mathfrak{g}_\alpha$, let x_α be the unique positive multiple of X such that $\langle x_\alpha, x_\alpha \rangle = \frac{2}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)}$.

let $y_\alpha := -\Theta(x_\alpha)$. If

$h_\alpha := \frac{2 H_\alpha}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} \Rightarrow$

$[x_\alpha, y_\alpha] = h_\alpha$, $[h_\alpha, x_\alpha] = 2x_\alpha$, $[h_\alpha, y_\alpha] = -2y_\alpha$

Corollary III.17 Given $\alpha \in \Sigma$, $X \in \mathfrak{g}_\alpha \setminus \{0\}$ and $x_\alpha, y_\alpha, h_\alpha$ as above $\Rightarrow$

$$\begin{array}{ccc} \mathfrak{sl}(2,\mathbb{R}) & \longrightarrow & \mathfrak{g} \\ e_+ & \longmapsto & x_\alpha \\ e_- & \longmapsto & y_\alpha \\ h & \longmapsto & h_\alpha \end{array}$$

is an injective Lie algebra homo with image

$\mathrm{span}\{x_\alpha, y_\alpha, h_\alpha\} =: \mathfrak{sl}(2,\mathbb{R})_\alpha \subset \mathfrak{g}$.

Rk $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in \mathfrak{a} \subset \mathfrak{p}$

Pf of Lemma III.16

$h_\alpha \in \mathfrak{a}$, $x_\alpha \in \mathfrak{g}_\alpha \Rightarrow$ $[h_\alpha, x_\alpha] = \alpha(h_\alpha) x_\alpha$. By defn. of h_α

$$\Rightarrow \alpha(h_\alpha) = \frac{2\alpha(H_\alpha)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} = \frac{2 B_{\mathfrak{g}}(H_\alpha, H_\alpha)}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} = 2$$

Same for $[h_\alpha, y_\alpha] = -2y_\alpha$.

To show that $[x_\alpha, y_\alpha] = h_\alpha$:

$[x_\alpha, y_\alpha] \overset{\text{def } y_\alpha}{=} [x_\alpha, -\Theta(x_\alpha)] = -[x_\alpha, \Theta(x_\alpha)]$

$h_\alpha \overset{\text{defn } h_\alpha}{=} \frac{2 H_\alpha}{B_{\mathfrak{g}}(H_\alpha, H_\alpha)} \overset{\text{defn } x_\alpha}{=} \langle x_\alpha, x_\alpha \rangle H_\alpha$

It will be enough to show:

Lemma III.18 Let $\alpha \in \Sigma$, $X \in \mathfrak{g}_\alpha \Rightarrow$

$\Rightarrow [X, \Theta(X)] = -\langle X, X \rangle H_\alpha$

Pf $-\langle X, X\rangle H_\alpha \in \mathfrak{a}$

First step: $[X, \Theta(X)] \in \mathfrak{a} = \mathfrak{g}_0 \cap \mathfrak{p}$ ✓

- $X \in \mathfrak{g}_\alpha \Rightarrow \Theta(X) \in \mathfrak{g}_{-\alpha} \Rightarrow$

  $\Rightarrow [X, \Theta(X)] \in \mathfrak{g}_{\alpha-\alpha} = \mathfrak{g}_0$
- $\Theta([X, \Theta(X)]) = [\Theta(X), X] =$

  $= -[X, \Theta(X)] \in \mathfrak{p}$.

Second step $-\langle X, X\rangle H_\alpha =$

$= B_{\mathfrak{g}}(X, \Theta(X)) H_\alpha$

$1+2 \Rightarrow \underbrace{[X, \Theta(X)] - B_{\mathfrak{g}}(X, \Theta(X)) H_\alpha}$ $\in \mathfrak{a}$

Third step — is orthogonal to $\mathfrak{a}$

w.r.t. the inner product $\langle , \rangle \Rightarrow$

$\Rightarrow$ it is zero.

let $Y \in \mathfrak{a} \subset \mathfrak{p}$

$\langle [X, \Theta(X)], Y\rangle = -B_{\mathfrak{g}}([X, \Theta(X)], \Theta(Y))$

$= B_{\mathfrak{g}}([X, \Theta(X)], Y) =$

$= -B_{\mathfrak{g}}(\Theta(X), [\underset{\mathfrak{g}_\alpha}{X}, \underset{\mathfrak{a}}{Y}])$

$= -B_{\mathfrak{g}}(\Theta(X), -\alpha(Y) X)$

$= B_{\mathfrak{g}}(\Theta(X), X)\, \alpha(Y)$

$= B_{\mathfrak{g}}(\Theta(X), X)\, \underbrace{B_{\mathfrak{g}}(H_\alpha, Y)}_{\text{scalar prod.}}$

$= B_{\mathfrak{g}}(\Theta(X), X)(-\langle H_\alpha, \Theta(Y)\rangle)$

$= B_{\mathfrak{g}}(\Theta(X), X)\langle H_\alpha, Y\rangle$

$\Rightarrow \langle [X, \Theta(X)] - B_{\mathfrak{g}}(\Theta(X), X) H_\alpha, Y\rangle = 0$

$\forall Y \in \mathfrak{a}$ $\blacksquare$

Remark $\alpha(R_\alpha) = 2$

$\alpha(H) = B_{\mathfrak{g}}(H, H_\alpha)$

Thm III.19 (1) Every f.d. rep. of $\mathfrak{sl}(2, \mathbb{R})$ is a direct sum of irred. repres.

(2) Every f.d. repr. of $\mathfrak{sl}(2, \mathbb{R})$ is classified, up to isom., by its dimension. If $\rho: \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{gl}(V)$ is an irred. rep. $\Rightarrow \rho(h)$ is diagonalizable with eigenvalues

$\{(\dim V - 1) - 2n : n = 0, \dots, \dim V - 1\}$

$= \{1 - \dim V, 3 - \dim V, \dots, -3 + \dim V, \dim V - 1\}$