Lecture

19 May 2021

Thm III.19 (1) every f.d. rep. of $SL(2,\mathbb{R})$ is completely reducible

(2) All f.d. irred rep. of $SL(2,\mathbb{R})$ $\rho: SL(2,\mathbb{R}) \to \mathfrak{gl}(V)$ are isomorphic. Moreover $\rho(h)$ is diagonaliz. with eigenvalues $\{1-\dim V, 3-\dim V, \dots, \dim V - 3, \dim V - 1\}$.

Example (1) $\dim V = 1 \Rightarrow \rho: SL(2,\mathbb{R}) \to \mathfrak{gl}(\mathbb{R})$.

$\rho(X) \equiv 0$; $\rho(h) = 0$, e.values $1-1=0$

(2) $\dim V = 2 \Rightarrow \rho: SL(2,\mathbb{R}) \to \mathfrak{gl}(\mathbb{R}^2)$

$\rho(X) := X$; $\rho(h) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, e.values $1-2$, $2-1$.

(3) $\dim V = 3 \Rightarrow \mathrm{ad}: \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(\mathfrak{sl}(2,\mathbb{R}))$ is irreducible and $\mathrm{ad}(h)$ has e.values $-2, 0, 2$.

(4) $\dim V = n+1 \Rightarrow V_n :=$ v.s. of homog. poly in X, Y of degree n =

$= \left\{ \sum_{k=0}^{n} a_k X^k Y^{n-k} : a_k \in \mathbb{R} \right\}$

$SL(2,\mathbb{R}) \curvearrowright V_n$ $(\rho(g)P)(X,Y) :=$

$= P((X,Y)g)$

$\rho: SL(2,\mathbb{R}) \to GL(V_n) \rightsquigarrow$

$\rightsquigarrow d_I\rho: \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(V_n)$

$(d_I\rho)(A)\, P(X,Y) =$

$= \frac{d}{dt}\Big|_{t=0} \rho(\exp tA) P(X,Y) =$

$= \frac{d}{dt}\Big|_{t=0} P((X,Y)\exp tA) \overset{\text{chain}}{\underset{\text{rule}}{=}}$

$(t \mapsto (X,Y)\exp tA \to P(\quad))$

$= d_{(X,Y)} P((X,Y)\cdot A)$

If $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}) \Rightarrow$

$\Rightarrow (d_I\rho)(A)\, P(X,Y) =$

$= \frac{\partial P}{\partial X}(X,Y)(aX + cY) +$

$+ \frac{\partial P}{\partial Y}(X,Y)(bX - cY)$

If $A = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow$

$\Rightarrow (d_I\rho)(h)\, X^k Y^{n-k} = (2k-n) X^k Y^{n-k}$

$\Rightarrow$ e.values $\{2k-n, k = 0, \dots, n\} =$

$= \{-n, 2-n, \dots, n-2, n\}$ with

(Recall that $\dim V_n = n+1$.)

e. spaces $\mathbb{R}Y^n, \mathbb{R}XY^{n-1}, \dots, \mathbb{R}X^{n-1}Y, \mathbb{R}X^n$ ☑

How to use this: If $\mathfrak{g}_\alpha \neq 0$, $\alpha \in \Sigma$, let $X \in \mathfrak{g}_\alpha$ and normalize X to get $X_\alpha \in \mathfrak{g}_\alpha$, $Y_\alpha \in \mathfrak{g}_{-\alpha}$, $h_\alpha \in \mathfrak{g}_0$ and $\mathrm{span}\{X_\alpha, Y_\alpha, h_\alpha\} \cong \mathfrak{sl}(2,\mathbb{R})_x$

Thm III.12 $\mathfrak{a} \subset \mathfrak{p}$ max. Abelian, $\Sigma \subset \mathfrak{a}^*$

(1) Σ spans $\mathfrak{a}^*$

(2) $\forall\, \alpha, \beta \in \Sigma$ $\quad \beta - \frac{2 B_\theta(H_\alpha, H_\beta)}{B_\theta(H_\alpha, H_\alpha)} \alpha \in \Sigma$

(3) $\frac{2 B_\theta(H_\alpha, H_\beta)}{B_\theta(H_\alpha, H_\alpha)} \in \mathbb{Z}$,

(*) $\boxed{\begin{array}{l} \mathfrak{a}^* \to \mathfrak{a} \\ \alpha \mapsto H_\alpha \\ \alpha(H) = B_\theta(H, H_\alpha) \end{array}}$

Pf If $\alpha, \beta \in \Sigma$, let

$W_{\alpha,\beta} := \sum^{\oplus} \{\mathfrak{g}_{\beta + k\alpha} : k \in \mathbb{Z}\} \subset \mathfrak{g}$

Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta} \Rightarrow$

$\Rightarrow \mathfrak{sl}(2,\mathbb{R})_x \curvearrowright W_{\alpha,\beta}$.

Moreover if $Y \in \mathfrak{g}_{\beta + k\alpha} \Rightarrow$

$\Rightarrow \mathrm{ad}_{\mathfrak{g}}(h_\alpha)(Y) = [h_\alpha, Y] =$

$= (\beta + k\alpha)(h_\alpha)\, Y \Rightarrow$ the e.values of $\mathrm{ad}_{\mathfrak{g}}(h_\alpha)$ on $W_{\alpha,\beta}$ are

$\{\beta(h_\alpha) + k\, \underbrace{\alpha(h_\alpha)}_{= 2} = \beta(h_\alpha) + 2k : k \in \mathbb{Z}, \beta + k\alpha \in \Sigma \cup \{0\}\}$

$r := \min\{k \in \mathbb{Z} : \beta + k\alpha \in \Sigma \cup \{0\}\}$

$s := \max\{k \in \mathbb{Z} : \beta + k\alpha \in \Sigma \cup \{0\}\}$.

let $n_x := n_x(\alpha, \beta)$ the max. dim. of an irred. submodule of $W_{\alpha,\beta}$ that is $\mathfrak{sl}(2,\mathbb{R})_x$-invariant.

By Thm III.19 $\Rightarrow$

$\left.\begin{array}{l} 1 - n_x = \beta(h_\alpha) + 2r \\ n_x - 1 = \beta(h_\alpha) + 2s \end{array}\right\} \Rightarrow$

$\Rightarrow -\beta(h_\alpha) = r + s \in \mathbb{Z}$

But $\beta(h_\alpha) \underset{\text{defn } h_\alpha}{=} \beta\left(\frac{2H_\alpha}{B_{\mathfrak{g}}(H_\alpha,H_\alpha)}\right) =$

$$= 2\,\frac{\beta(H_\alpha)}{B_{\mathfrak{g}}(H_\alpha,H_\alpha)} \overset{(*)}{=} 2\,\frac{B_{\mathfrak{g}}(H_\alpha,H_\beta)}{B_{\mathfrak{g}}(H_\alpha,H_\alpha)}$$

Again by the theory of irr. f.d. rep. of $\mathfrak{sl}(2,\mathbb{R})$ $\Rightarrow$ $\mathfrak{g}_{\beta+k\alpha}\neq\{0\}$ if $r\le k\le s$. Since $\beta\in\Sigma$ $\Rightarrow$

$\mathfrak{g}_\beta\neq(0)$ $\Rightarrow$ $r\le 0\le s$

Since $-\beta(h_\alpha)=r+s$ $\Rightarrow$

$\Rightarrow$ $r\le-\beta(h_\alpha)\le s$ $\Rightarrow$

$\Rightarrow$ $\mathfrak{g}_{\beta-\beta(h_\alpha)\alpha}\neq(0)$ Two posib:

(1) $\beta-\beta(h_\alpha)\alpha\neq0$ $\Rightarrow$ $\beta-\beta(h_\alpha)\alpha\in\Sigma$

(2) $\beta-\beta(h_\alpha)\alpha=0$ $\Rightarrow$ $\beta=\beta(h_\alpha)\alpha$

$\Rightarrow$ $\beta(h_\alpha)=\beta(h_\alpha)\underbrace{\alpha(h_\alpha)}_{=2}$

$\Rightarrow$ $\beta(h_\alpha)=0$ $\Rightarrow$ $\beta-\beta(h_\alpha)\alpha=$
$=\beta\in\Sigma$.

$\Rightarrow$ $\beta-\beta(h_\alpha)\alpha\in\Sigma$. $\blacksquare$

III.4 Abstract root systems

$\mathbb{E}$ Euclidean space, $\langle\ ,\ \rangle$ inner product, $\gamma\in\mathbb{E}\setminus\{0\}$

$\sigma_\gamma(\alpha):=\alpha-\frac{2\langle\alpha,\gamma\rangle}{\langle\gamma,\gamma\rangle}\gamma$ is the refl. in $\mathbb{E}$ w.r. to the hyperplane $\gamma^\perp$. $\sigma_\gamma(\gamma)=-\gamma$.

Defn A root system of rank $\dim\mathbb{E}$ is a subset $\Sigma\subset\mathbb{E}\setminus\{0\}$ s.t.

(1) Σ spans $\mathbb{E}$

(2) $\sigma_\alpha(\Sigma)=\Sigma$ $\forall\alpha\in\Sigma$

(3) $2\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\mathbb{Z}$ $\forall\ \alpha,\beta\in\Sigma$

Ex. The root spaces of $\mathfrak{a}$ in $\mathfrak{g}$.

Rk (3) $\Rightarrow$ if $\beta=\lambda\alpha\in\Sigma$ then $\lambda\in\{\pm\frac12,\pm1,\pm2\}$ (because $2\lambda\in\mathbb{Z}$ and $\frac{2}{\lambda}\in\mathbb{Z}$)

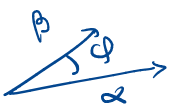
Defn. A root system Σ is reduced if from $\beta=\lambda\alpha\in\Sigma$ $\Rightarrow$ $\lambda=\pm1$.

If Σ is any root system $\Rightarrow$ $\Sigma'=\{\alpha\in\Sigma:\ \frac{\alpha}{2}\notin\Sigma\}$

Want to understand config. of root systems. Say $\varphi:=\measuredangle(\alpha,\beta)$

$$n(\beta,\alpha):=2\frac{\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}=\frac{2\|\beta\|}{\|\alpha\|}\cos\varphi$$

(3) $\Rightarrow$ $n(\beta,\alpha)=\frac{2\|\beta\|}{\|\alpha\|}\cos\varphi\in\mathbb{Z}$



$n(\alpha,\beta)=\frac{2\|\alpha\|}{\|\beta\|}\cos\varphi\in\mathbb{Z}$

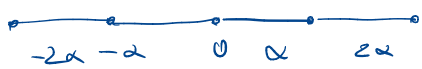
$\Rightarrow$ $4\cos^2\varphi\in\mathbb{Z}$ and $n(\alpha,\beta)\,n(\beta,\alpha)>0$

$4\cos^2\varphi\in\mathbb{Z}$; $\overbrace{n(\beta,\alpha)\in\mathbb{Z},\ n(\alpha,\beta)\in\mathbb{Z}}$

φ	$n(\beta,\alpha)$	$n(\alpha,\beta)$	$\|\beta\|^2/\|\alpha\|^2$
$\pi/2$	0	0	undetermined
$\pi/3$	1	1	1
$2\pi/3$	-1	-1	1
$\pi/4$	2	1	2
$3\pi/4$	-2	-1	2
$\pi/6$	3	1	3
$5\pi/6$	-3	-1	3

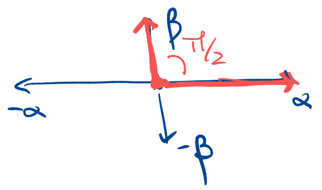
Examples

- System of rank 1

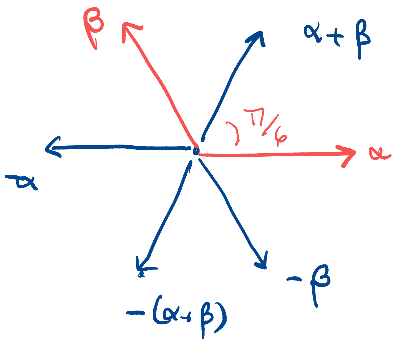


reduced

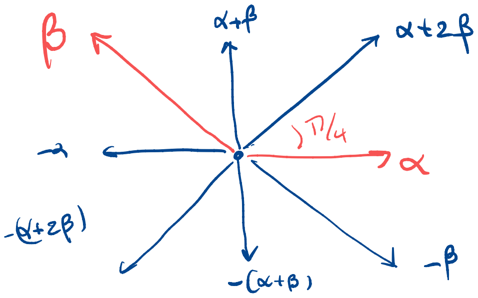
- **Systems of rank 2**



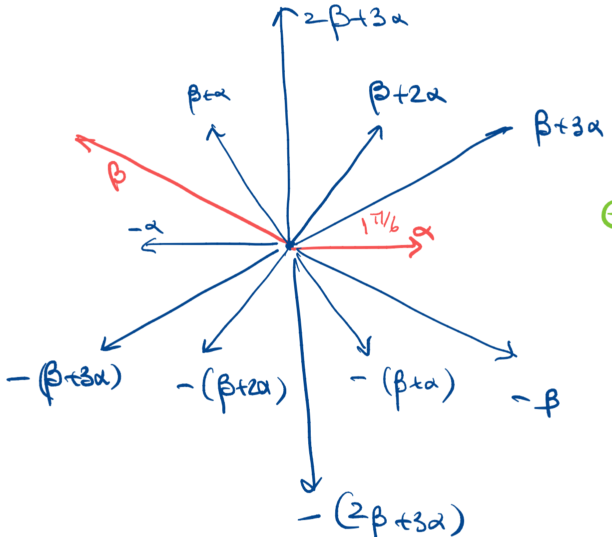
$A_1 \times A_1$



A_2



B_2



G_2

Lemma III.20 $\alpha, \beta \in \Sigma$ not proportional

- $\langle \alpha, \beta \rangle > 0 \Rightarrow \beta - \alpha \in \Sigma$
- $\langle \alpha, \beta \rangle < 0 \Rightarrow \alpha + \beta \in \Sigma$

Defn A basis of Σ is a subset $\Delta \subset \Sigma$ s.t. (1) Δ is a basis of $\mathbb{E}$
(2) If $\alpha \in \Sigma$, $\alpha = \sum_{\beta \in \Delta} c_\beta \beta \Rightarrow$ all c_β have the same sign.

Rk For a fixed Δ, one can write

$$\Sigma = \underset{c_\beta > 0}{\Sigma^+} \sqcup \underset{c_\beta < 0}{\Sigma^-}$$

The elements of Δ are called simple roots.

Lemma III.21 $\forall \alpha, \beta \in \Delta, \alpha \neq \beta$
$\Rightarrow \langle \alpha, \beta \rangle < 0$.

Pf If $\langle \alpha, \beta \rangle > 0 \overset{\text{III.20}}{\Rightarrow}$
$\Rightarrow \beta - \alpha \in \Sigma$ $\lightning$ $\blacksquare$

Notation $\alpha \in \mathbb{E} \setminus \{0\}$,

$$P_\alpha = \{\beta \in \mathbb{E} : \langle \alpha, \beta \rangle > 0\}.$$

Defn • A Weyl chamber is a conn. cpt. of $\mathbb{E} \setminus \bigcup_{\alpha \in \Sigma} P_\alpha$

- An element $\gamma \in \mathbb{E}$ is regular if $\langle \alpha, \gamma \rangle \neq 0$ $\forall \alpha \in \Sigma$, that is if γ belongs to some Weyl chamber.

Exercise Check that the defn. are compatible. $(\mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h})$.