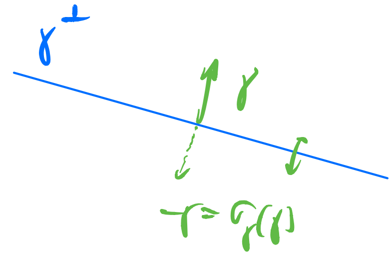
Exercise Class Symmetric Spaces 20.5.21

I) Recap: Root Systems:

$\mathbb{E}$ Euclidean space, $\langle\ ,\ \rangle$ inner product, $\gamma \in \mathbb{E}\setminus\{0\}$



$$\sigma_\gamma(\alpha) := \alpha - \frac{2\langle\alpha,\gamma\rangle}{\langle\gamma,\gamma\rangle}\gamma$$ is the refl. in $\mathbb{E}$ w.r.t. the hyperplane $\gamma^\perp$. $\sigma_\gamma(\gamma) = -\gamma$.

Defn A root system of rank $\dim \mathbb{E}$ is a subset $\Sigma \subset \mathbb{E}\setminus\{0\}$ s.t.

(1) Σ spans $\mathbb{E}$

(2) $\sigma_\alpha(\Sigma) = \Sigma \quad \forall \alpha \in \Sigma$

(3) $2\dfrac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z} \quad \forall\, \alpha,\beta \in \Sigma$

Most important example for us:

$(\mathfrak{g},\theta)$ OSLA, $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$,

$\mathfrak{a} \subseteq \mathfrak{p}$ maximal abelian,

$\langle X, Y\rangle := -B_{\mathfrak{g}}(X, \theta(Y))$

$\mathfrak{a}$ abelian $\Rightarrow \{\mathrm{ad}(H) : H \in \mathfrak{a}\}$ commute

$\Rightarrow$ Simultaneously diagonalize $\mathrm{ad}(H): \mathfrak{g} \to \mathfrak{g}$.

$$\rightsquigarrow \quad \mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid \mathrm{ad}(H) X = \alpha(H) \cdot X \quad \forall H \in \mathfrak{a} \}$$
$\alpha \in \mathfrak{a}^*$

"generalized eigenspaces"

$$\Sigma := \{ \alpha \in \mathfrak{a}^* \mid \mathfrak{g}_\alpha \neq (0) \}. \subset \mathfrak{a}^*$$

Identify $\mathfrak{a}^* \longrightarrow \mathfrak{a}$ via $\langle \cdot , \cdot \rangle$

$\rightsquigarrow$ $\langle H_\alpha, H \rangle := \alpha(H) \quad \forall H \in \mathfrak{a}$ defines H_α.

Then $\Sigma \cong \{ H_\alpha \mid \alpha \in \mathfrak{a} \} \subseteq \mathfrak{a} \cong E$

<u>root system</u>.

II) Exercise Sheet 5 20.5.21

Exercise 1.(Maximal abelian subspaces and regular elements in $\mathfrak{sl}(n,\mathbb{R})$):

Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{R})$. A Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where $\mathfrak{p} = \{X \in \mathfrak{sl}(n,\mathbb{R}) : X = X^t\}$ and $\mathfrak{k} = \{X \in \mathfrak{sl}(n,\mathbb{R}) : X = -X^t\}$. Define

$$\mathfrak{a} = \left\{ \operatorname{diag}(t_1,\dots,t_n) : t_j \in \mathbb{R}, \sum_{j=1}^n t_j = 0 \right\}.$$

a) Prove that $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$.

Sol: ∘ $\mathfrak{a}$ abelian ✓

∘ $\mathfrak{a}$ maximal: Let $\mathfrak{a} \subseteq \mathfrak{a}' \subseteq \mathfrak{p}$ be abelian.

Then:

$$0 = [X, Y] \qquad \forall X \in \mathfrak{a} \ \forall Y \in \mathfrak{a}'$$

Write $Y = (y_{ij})_{ij}$ & $X = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \in \mathfrak{a}$.

$$0 = [X, Y]$$

$$= \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{pmatrix} - \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & & \vdots \\ y_{n1} & \cdots & y_{nn} \end{pmatrix} \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix}$$

$$= (t_i y_{ij} - y_{ij} \cdot t_j)_{ij}$$

$$\Longrightarrow \quad t_i \cdot y_{ij} = t_j \cdot y_{ij}$$

For $t_1, \dots, t_n$ pw. dist. $\quad y_{ij} = 0 \quad \forall i \neq j$.

$\Longrightarrow$ Y is diagonal hence $Y \in \mathfrak{a}$

$\Longrightarrow$ $\mathfrak{a}' = \mathfrak{a}$. $\square$

b) Prove (without appealing to the general theorem) any maximal abelian subspace of $\mathfrak{p}$ is of the form $S\mathfrak{a}S^{-1}$ where $S \in SO(n)$.

Sol: Let $\mathfrak{a}' \subset \mathfrak{p}$ be maximal abelian & $\{Y_1, \dots, Y_r\}$ a basis of $\mathfrak{a}'$. Note: $[Y_i, Y_j] = 0 \ \forall i,j$!

$\Longrightarrow$ $\{Y_1, \dots, Y_r\}$ commute & symmetric!

$\Longrightarrow$ Simultaneously diagonalizable: $\exists S_0 \in O(n)$:

$$S_0 Y_i S_0^{-1} = D_i \text{ diagonal } \forall i = 1, \dots, r.$$

If $S_0 \in O(n) \setminus SO(n)$ then set $S := \underbrace{\begin{pmatrix} -1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}}_{=: J} \cdot S_0$

else $S = S_0$.

In the first case: $S \cdot Y_i \cdot S^{-1} = J \cdot S_0 Y_i S_0^{-1} J^{-1}$

$= J D_i J^{-1} = D_i'$. is still diagonal. ✓

$\Longrightarrow$ $\underbrace{S \cdot \mathfrak{a}' S^{-1}}_{\text{max. abelian}} \subseteq \mathfrak{a}$

$\xRightarrow{\text{maximality}}$ $S \mathfrak{a}' \cdot S^{-1} = \mathfrak{a}$. $\square$

c) Show that $X \in \mathfrak{p}$ is a regular element if and only if all of its eigenvalues are distinct.

<u>Sol:</u> Let $X \in \mathfrak{p}$ be regular, i.e. $C_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is maximal abelian.

By b): $\exists S \in SO(n)$: $\mathfrak{a} = S(C_{\mathfrak{g}}(X) \cap \mathfrak{p})S^{-1}$

$$= C_{\mathfrak{g}}(SXS^{-1}) \cap \mathfrak{p} \quad (*)$$

$$\Rightarrow \quad SXS^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} =: D$$ with eigenvalues $\lambda_1 - \lambda_n$ of X.

Let P_{ij} be the permutation matrix that only exchanges e_i with e_j.

$$P_{ij} = \begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & 1 & & \\ & & & & 1 & & & \\ & & & & & \ddots & & \\ & & & & & & 1 & \\ & & & 1 & & & 0 & \\ & & & & & & & 1 \\ & & & & & & & & \ddots \\ & & & & & & & & & 1 \end{bmatrix} \in \mathfrak{p}$$ symmetric

Then $[D, P_{ij}](e_k) = D P_{ij}(e_k) - P_{ij} D(e_k) = 0 \quad \forall k \notin \{i,j\}$

$$[D, P_{ij}](e_i) = (\lambda_j - \lambda_i)e_j$$

$$[D, P_{ij}](e_j) = (\lambda_i - \lambda_j)e_i$$

If $\lambda_i = \lambda_j$ for some $i \neq j$, then

$P_{ij} \in C_{\mathfrak{g}}(D) \cap \mathfrak{p}$ but $P_{ij} \notin \mathfrak{a}$ contradicting $(*)$.

$\lightning$.

Vice Versa: Let $X \in \mathfrak{p}$ with dist. e.v.

$\Longrightarrow$ $\exists$ max ab. $\mathfrak{a}' \ni X$ & $S \in SO(n)$:

$S \mathfrak{a}' S^{-1} = \mathfrak{a}$

$$\Longrightarrow \quad S X S^{-1} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} =: D$$

$\xRightarrow[\text{pw. dist.}]{\lambda_1, \dots, \lambda_n}$ $C_{\mathfrak{g}}(D) \cap \mathfrak{p} = \mathfrak{a}$ as in (a)!

$\square$

Exercise 2.(Maximal abelian subspaces and regular elements in $\mathfrak{sp}(2n,\mathbb{R})$):

Let $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{R})$. Recall that a Cartan decomposition of $\mathfrak{g}$ is given by $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ where

$$\mathfrak{p} = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} : A = A^t, B = B^t \right\}$$

and

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} : A = -A^t, B = B^t \right\}.$$

a) Define

$$\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} : A = \mathrm{diag}(t_1,\ldots,t_n). \right\}.$$

Prove that A is a maximal abelian subspace of $\mathfrak{p}$.

<u>Sol:</u> ∘ $\mathfrak{a}$ abelian: ✓

∘ $\mathfrak{a}$ is maximal: Suppose $\mathfrak{a} \subseteq \mathfrak{a}' \subseteq \mathfrak{p}$ abelian.

Let $X = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \in \mathfrak{a}$ & $Y = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathfrak{a}'$.

Then $0 = [Y, X] = \begin{pmatrix} [A,D] & -BD-DB \\ BD+DB & [A,D] \end{pmatrix}$

As in 1(a): A is diagonal

$BD + DB = 0$ $\qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

$\Leftrightarrow \quad b_{ij}(\lambda_i + \lambda_j) = 0 \quad \forall i,j \quad \forall D!$

$\Rightarrow \quad b_{ij} = 0 \Rightarrow Y \in \mathfrak{a} \Rightarrow \mathfrak{a} = \mathfrak{a}'.$

b) Show that $X \in \mathfrak{p}$ is a regular element if and only if all of its eigenvalues are distinct and non-zero.

Sol: Let $X \in \mathfrak{p}$ be regular, i.e. $C_{\mathfrak{g}}(X) \cap \mathfrak{p}$ is max. abelian

$\Rightarrow \exists\ k \in K = SO(2n, \mathbb{R}) \cap Sp(2n, \mathbb{R})$ s.t.

$$k(C_{\mathfrak{g}}(X) \cap \mathfrak{p})k^{-1} = C_{\mathfrak{g}}(kXk^{-1}) \cap \mathfrak{p} = \mathfrak{a}$$

Write

$$kXk^{-1} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix}$$

with $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ & ev's $\lambda_1, \ldots, \lambda_n$

As for $\mathfrak{sl}_n(\mathbb{R})$:

$$\begin{pmatrix} P_{ij} & 0 \\ 0 & -P_{ij} \end{pmatrix} \in C_{\mathfrak{g}}(X) \cap \mathfrak{p} \quad \text{if } \lambda_i = \lambda_j \text{ for } i \neq j.$$

but $\notin \mathfrak{a}$

If $\lambda_i = 0$ then $\begin{pmatrix} 0 & E_{ii} \\ E_{ii} & 0 \end{pmatrix} \in C_{\mathfrak{g}}(X) \cap \mathfrak{p}$

but $\notin \mathfrak{a}$. $\lightning$

Vice versa: Let $X \in \mathfrak{p}$ have dist. non-zero e.v.'s

$\Rightarrow$ $X \in \mathfrak{a}'$ max. ab. & $\exists\, k \in K$: $k\mathfrak{a}'k^{-1} = \mathfrak{a}$

$$\Rightarrow \quad kXk^{-1} = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} =: T$$

Clearly, $C_{\mathfrak{g}}(T) \cap \mathfrak{p} \supseteq \mathfrak{a}$. For $Y \in C_{\mathfrak{g}}(T) \cap \mathfrak{p}$ we check as in a) that $Y_{ij} = 0$ $\forall i \neq j$.

$\square$

Exercise 4.(Irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$):

Let $V = \mathbb{C}[X,Y]$ be the vector space of polynomials in two variables. Let V_m denote the vector subspace of all homogeneous polynomials of degree m. This has a basis given by the monomials $X^m, X^{m-1}Y, \ldots, Y^m$. We turn this vector subspace into a module for $\mathfrak{sl}(2,\mathbb{C})$ by defining a Lie algebra homomorphism $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(V_m)$ in the following way

$$\phi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = X\frac{\partial}{\partial Y}, \quad \phi\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = Y\frac{\partial}{\partial X}, \quad \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Show that this defines an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$.

Sol: Set $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

& $E' = X\frac{\partial}{\partial Y}$, $F' = Y\frac{\partial}{\partial X}$, $H' = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}$.

Check: $[E,F] = H$, $[E,H] = -2E$, $[F,H] = 2F$,

& $[E',F'] = H'$, $[E',H'] = -2E'$, $[F',H'] = 2F'$

e.g.: $[E',F'] = X\frac{\partial}{\partial Y}\left(Y\frac{\partial}{\partial X}\right) - Y\frac{\partial}{\partial X}\left(X\frac{\partial}{\partial Y}\right)$

$= X\cdot\frac{\partial}{\partial X} + \cancel{XY\frac{\partial^2}{\partial Y\partial X}} - Y\frac{\partial}{\partial Y} - \cancel{YX\frac{\partial^2}{\partial X\partial Y}}$

$= H'$.

$\Rightarrow$ φ is a Lie alg. hom / repr.

φ is irreducible: Suppose $0 \subsetneq V' \subsetneq V_m$ is an invariant subspace. Take $0 \neq v' \in V'$.

Then $\deg_y(E'v) < \deg_y(v) \quad \forall v \in V_m$

$\Rightarrow$ $\exists$ minimal $k \in \mathbb{N}$: $(E')^{k+1} v' = 0$.

$\Rightarrow$ $0 \neq (E')^k v \in \ker(E') = \mathbb{C} \cdot X^m$, i.e.

$(E')^k v' = \alpha \cdot X^m$ for some $\alpha \in \mathbb{C}^\times$.

Repeatedly applying F' we get a full basis for V_m:

$$
\begin{aligned}
F'(X^m) &= X^{m-1} y \\
(F')^2(X^m) &= X^{m-2} y^2 \\
&\vdots \\
& \quad X y^{m-1} \\
& \quad y^m
\end{aligned}
$$

By inv. $\Rightarrow$ $\{X^m, X^{m-1}y, \ldots, X y^{m-1}, y^m\} \subseteq V'$

$\Rightarrow$ $V' = V_m$. $\square$

Exercise 3.(The Siegel upper half space):

Let

$$\mathbb{H}_n := \{Z \in \mathbb{C}^{n\times n} : Z = Z^t,\ \mathrm{Im}(Z) \text{ is positive-definite}\}.$$

Find an explicit isomorphism between $\mathbb{H}_n$ and $\mathrm{Sp}(2n,\mathbb{R})/(\mathrm{SO}(2n) \cap \mathrm{Sp}(2n,\mathbb{R}))$. Use this and the previous exercise to construct a maximal flat of $\mathbb{H}_n$.

<u>Hint:</u> : Consider the map

$$\phi : \mathrm{Sp}(2n,\mathbb{R}) \to \mathbb{H}_n,$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto (Ai + B)\cdot(Ci + D)^{-1}.$$

Compare to Möb. trafo.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot i = \dfrac{ai+b}{ci+d}$

<u>Sol:</u> <u>Claim:</u> $Sp(2n,\mathbb{R}) \curvearrowright \mathbb{H}_n$ via generalized Möbius transformations:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z := (A\cdot Z + B)(CZ + D)^{-1}$$

The action is transitive with

$$\mathrm{stab}(i\cdot I_n) = K := Sp(2n,\mathbb{R}) \cap SO(2n,\mathbb{R}).$$

<u>Pf:</u> • <u>Well-definedness:</u> $CZ+D$ invertible & $g\cdot Z$ symm. & $\mathrm{Im}(g\cdot Z)$ pos.-def.

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad Z \in \mathbb{H}_n.$$

Write $P = AZ + B$, $Q = CZ + D$.

$$g \in Sp(2n, \mathbb{R}) \iff A^tC = C^tA, \quad B^tD = D^tB, \quad A^tD - C^tB = I$$

"$ad - bc = 1$"

$$P^t\overline{Q} - Q^t\overline{P} = (ZA^t + B^t)(C\overline{Z} + D) - (ZC^t + D^t)(A\overline{Z} + B)$$

$$= ZA^tC\overline{Z} + ZA^tD + B^tC\overline{Z} + B^tD - (ZC^tA\overline{Z} + ZC^tB + D^tA\overline{Z} + D^tB)$$

$$= Z - \overline{Z} = 2i \cdot \mathrm{Im}(Z)$$

If $\xi \in \ker Q$ then:

$$\xi^t \cdot \mathrm{Im}(Z) \cdot \xi = \frac{1}{2i}\left(\xi^t P^t \overline{Q}\xi - \xi^t Q^t P \xi\right) = 0$$

$$\overset{\mathrm{Im} Z \gg 0}{\Longrightarrow} \quad \xi = 0 \implies \ker Q = (0) \implies CZ + D \text{ invertible.}$$

$$M = g \cdot Z \text{ symmetric} \iff M^t = M \iff \underbrace{(PQ^{-1})^t}_{= (Q^{-1})^t P^t} = PQ^{-1}$$

$$\iff P^tQ = Q^tP.$$

$$\iff (Z^tA^t + B^t)(CZ + D) = (Z^tC^t + D^t)(AZ + B)$$

Check again by using , and .

Similarly, $\operatorname{Im}(g\cdot Z)$ is positive def.

We'll skip the verification that $G \curvearrowright \mathbb{H}_n$ is an action: $g\cdot(h\cdot Z) = (g\cdot h)\cdot Z$

<u>Transitive:</u> Let $Z = X + i\cdot Y \in \mathbb{H}_n$

Set $g := \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y}^{-1} \end{pmatrix}$, $h = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \in G$

Then check: $(hg)\cdot iI = X + i\cdot Y$.

<u>Stabilizer:</u> Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ s.t. $g\cdot(iI) = iI$.

$\Leftrightarrow$ $iI = (A\cdot i + B)(C\,i + D)^{-1}$

$\Leftrightarrow$ $-C + i\cdot D = B + A\cdot i$

$\Leftrightarrow$ $B = -C$, $A = D$.

Because g is symplectic:

$I = AA^t + BB^t$ & $A^tB = B^tA$.

$$g^tg = \begin{pmatrix} A^tA + B^tB & A^tB - B^tA \\ B^tA - A^tB & B^tB + A^tA \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

$\Rightarrow$ $g \in K = Sp(2n, \mathbb{R}) \cap SO(2n, \mathbb{R})$.

$\overset{?}{\underset{a}{\Leftarrow}}$: $K \subset \mathrm{stab}(i \cdot I)$; see solution.

$\rightsquigarrow \quad \varphi : G/K \xrightarrow{\sim} \mathbb{H}_n$

$\qquad gK \longmapsto g \cdot (iI)$

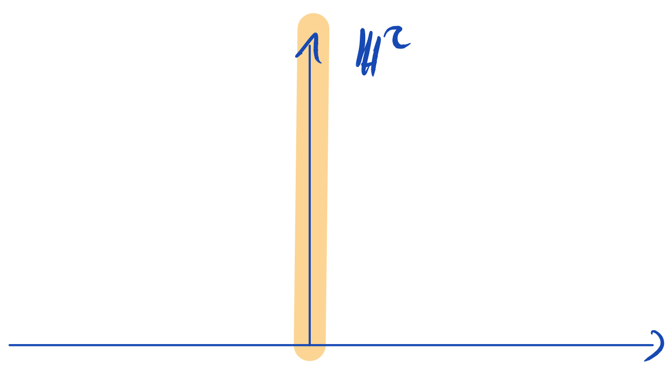
By ex (2): $\mathfrak{a} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \;\middle|\; A = \mathrm{diag}(t_1, \dots, t_n) \right\}$.

is max. abelian.

Thus: $\exp(\mathfrak{a}) = \left\{ \begin{pmatrix} \lambda_1 & & & & & 0 \\ & \ddots & & & & \\ & & \lambda_n & & & \\ & & & \lambda_1^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & \lambda_n^{-1} \end{pmatrix} \;\middle|\; \lambda_1, \dots, \lambda_n > 0 \right\}$

$$\varphi(\exp(\mathfrak{a})K) = \{ g \cdot iI \mid g \in \exp(\mathfrak{a}) \}$$

$$= \left\{ i \begin{pmatrix} \lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix} \;\middle|\; \lambda_i > 0 \right\}.$$



$\square$