lecture

26 May 2021

Defn A **root system of rank = dim $\mathbb{E}$** is a subset $\Sigma \subset \mathbb{E} \setminus \{0\}$ s.t.

(1) Σ spans $\mathbb{E}$

(2) If σ_α is the refl. w.r.t. the hyperplane $\perp \alpha \in \Sigma$, then $\sigma_\alpha(\Sigma) = \Sigma$

(3) $2\dfrac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z} \quad \forall \alpha, \beta \in \Sigma$

(4) Σ is reduced if $\alpha, \lambda\alpha \in \Sigma \Rightarrow$
$\Rightarrow \lambda = \pm 1$.

(5) A **basis** $\Delta \subset \Sigma$ is a subset s.t.

- Δ spans $\mathbb{E}$
- If $\alpha \in \Sigma$, $\alpha = \sum_{\beta \in \Delta} c_\beta \beta$
  $\Rightarrow c_\beta$ have the same sign.

$$n(\beta,\alpha) := 2\frac{\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$$

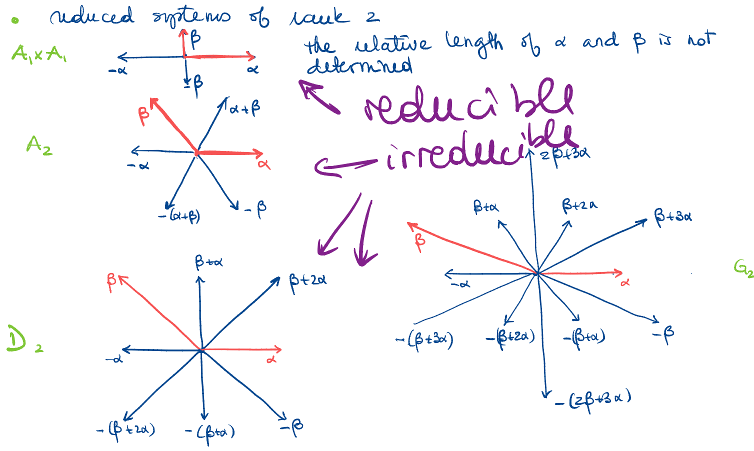
① φ	② $n(\beta,\alpha)$	② $n(\alpha,\beta)$	$\|\beta\|^2/\|\alpha\|^2$
$\pi/2$	0	0	undetermined
$\pi/3$	1	1	1
$2\pi/3$	-1	-1	1
$\pi/4$	2	1	2
$3\pi/4$	-2	-1	2
$\pi/6$	3	1	3
$5\pi/6$	-3	-1	3

A_1 

reduced

- reduced systems of rank 2

the relative length of α and β is not determined



Defn. If Σ is a root system, the **Cartan matrix** of Σ w.r.t. a basis Δ is $(n(\alpha,\beta))_{\alpha,\beta\in\Delta}$.

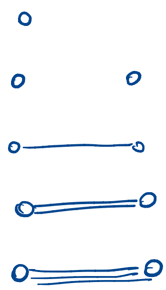
Ex $G_2 \Rightarrow \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

Fact A reduced root system is det. up to isom. by its Cartan matrix.

Defn (1) A **Coxeter graph** is a finite graph whose vertices are connected by 0, 1, 2 or 3 edges.

(3) A coxeter graph of a root system w.r.t. Δ has the elements of Δ as vertices and $n(\alpha,\beta)\,n(\beta,\alpha)$ edges between α and β.

Ex.
A_1
$A_1 \times A_1$
A_2
B_2
G_2



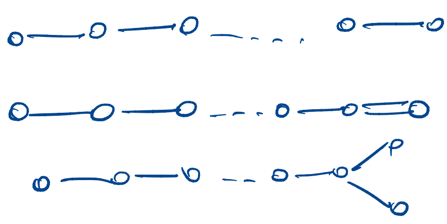
Defn. If $\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2$
$\Sigma \subset \mathbb{E}$ root system of $\mathbb{E} \Rightarrow$
$\Rightarrow \Sigma_i := \Sigma \cap \mathbb{E}_i$ is a root system in $\mathbb{E}_i \Rightarrow$
Σ is not irreducible.

Any root system can be decomposed into sum of irreducible.

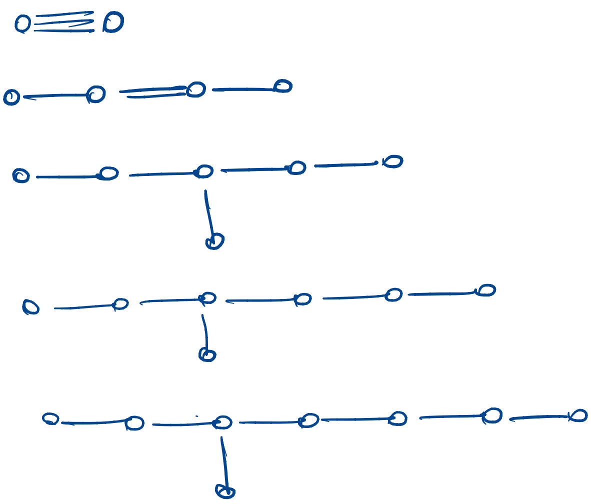
Σ is irreducible $\Leftrightarrow$ its Coxeter graph is connected

Thm III.22 Every conn. nonempty Coxeter graph of a root system is isom. to one of the following

A_n (n vert ≥ 1)
B_n ($n \geq 2$)
D_n ($n \geq 4$)

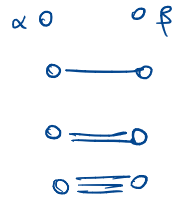


G_2
F_4
E_6
E_7
E_8



Idea of proof: Take a Cox. gr. G with vertex set Σ. ⇝ $(\ ,\ )$ bilinear form on $\mathbb{R}^{|\Sigma|}$ with basis $\{e_\alpha\}_{\alpha\in\Sigma}$

$$(e_\alpha, e_\beta) = \begin{cases} \cos\frac{\pi}{2} \\ \cos\frac{2\pi}{3} \\ \cos\frac{3\pi}{4} \\ \cos\frac{5\pi}{6} \end{cases}$$



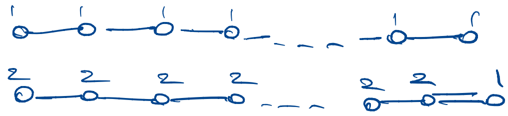
If Σ is irred. ⇒ ∃! inner product on $\mathbb{E}$ (up to a constant) ⇒ .... classif. ☒.

We still need the relative length of the roots to get the Cartan matrix. ⇒ we put weights on each vertex ∝ proportional to the length² (α,α) ⇝ Dynkin diagram of Σ.

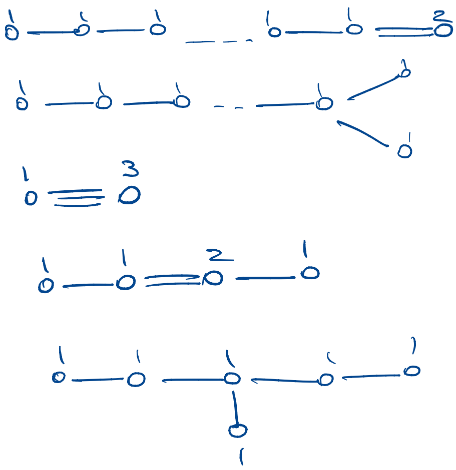
Two proportional Dynkin diagrams describe the same Cartan matrix.

Thm III.23 Each non-empty conn. Dynkin diagram is isom. to one of the following

A_n
B_n



C_n $(n\geq 3)$
D_n $(n\geq 4)$
G_2
F_4
E_6
E_7
E_8



Sometimes

$\mathfrak{g}$ simple Lie alg over $\mathbb{C}$ ⇝ irred. root system ⇝ Cartan matr. ⇝ Dynkin diagram

Dynkin diagr ⇝ Cartan matrix ⇝ irred. root system ⇝ $\mathfrak{g}$ simple Lie alg. over $\mathbb{C}$.

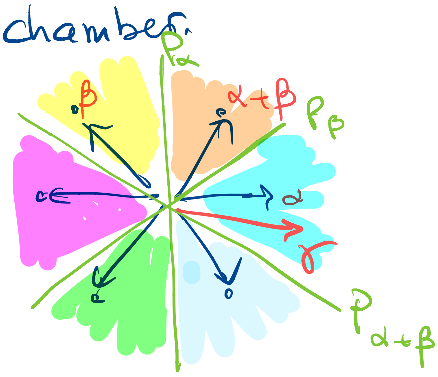
[Helgason, Chapter X]

Defn If $\alpha\in\mathbb{E}\setminus\{0\}$,

$$P_\alpha := \{\beta\in\mathbb{E} : \langle\alpha,\beta\rangle = 0\}$$

Then a Weyl chamber is a conn. comp. of $\mathbb{E}\setminus\bigcup_{\alpha\in\Sigma} P_\alpha$.

An elem. γ is regular if $\langle\alpha,\gamma\rangle\neq 0$ $\forall\alpha\in\Sigma$, that is if γ belongs to a Weyl chamber.



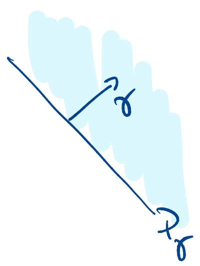
γ regular ⇝ $\Delta(\gamma)$ basis

γ,γ' regular in the same Weyl ch ⇒ $\Delta(\gamma)=\Delta(\gamma')$

$\gamma \in \mathbb{E}$ regular element

$\Sigma^+(\gamma) := \{\alpha\in\Sigma : \langle\alpha,\gamma\rangle > 0\} =$

= roots on the same half space as γ.



$$\Sigma = \Sigma^+(\gamma) \sqcup \Sigma^-(\gamma)$$

Defn. $\alpha\in\Sigma^+(\gamma)$ is indecomposable if it cannot be written as a sum of two elements in $\Sigma^+(\gamma)$.

$\Delta(\gamma) := \{\alpha\in\Sigma^+(\gamma) : \alpha \text{ indecomp.}\}$.

Thm III.24 If $\gamma\in\mathbb{E}$ is regular $\Rightarrow$ $\Delta(\gamma)$ is a basis and every basis of Σ is of this form.

Pf (1) $\mathbb{Z}_{\geq 0}$-span of $\Delta(\gamma)$ is $\Sigma^+(\gamma)$. In fact, let

$\{\langle\alpha,\gamma\rangle : \alpha\in\Sigma^+(\gamma)\} =$

$= \{0 < s_1 \leq \dots \leq s_e\}$

If $\langle\alpha,\gamma\rangle = s_1 \Rightarrow \alpha$ must be indecomposable, that is $\alpha\in\Delta(\gamma)$.

In fact if not, $\alpha = \beta_1+\beta_2$, $\beta_i\in\Sigma^+(\gamma)$. But $\langle\alpha,\gamma\rangle =$

$= \underbrace{\langle\beta_1,\gamma\rangle}_{>0} + \underbrace{\langle\beta_2,\gamma\rangle}_{>0}$ ↯

Assume that any $\alpha\in\Sigma^+(\gamma)$ with $\langle\alpha,\gamma\rangle = s_k$ is in the $\mathbb{Z}_{\geq0}$-span of $\Sigma^+(\gamma)$. Let $\alpha\in\Sigma^+(\gamma)$ with $\langle\alpha,\gamma\rangle = s_{k+1}$.

If α is indec. $\Rightarrow \alpha\in\Delta(\gamma)$.

If not $\alpha = \beta_1+\beta_2$ where $\langle\beta_i,\gamma\rangle < s_{k+1}$

$\Rightarrow \beta_i \in \mathbb{Z}_{\geq0}$-span of $\Sigma^+(\gamma)$

$\Rightarrow$ same true for α.

(2) $\Delta(\gamma)$ is linearly indep. In fact let $\sum_{\alpha\in\Delta(\gamma)}\lambda_\alpha\alpha = 0$.

Let $A = \{\alpha\in\Delta(\gamma) : \lambda_\alpha > 0\}$

$B = \{\beta\in\Delta(\gamma) : \lambda_\beta < 0\}$

$\sum_{\alpha\in\Delta(\gamma)}\lambda_\alpha\alpha = 0 \Rightarrow$

$\Rightarrow \sum_{\alpha\in A}\lambda_\alpha\alpha = -\sum_{\beta\in B}\lambda_\beta\beta$

$\Rightarrow \left\|\sum_{\alpha\in A}\lambda_\alpha\alpha\right\|^2 =$

$= -\left\langle \sum_{\alpha\in A}\lambda_\alpha\alpha, \sum_{\beta\in B}\lambda_\beta\beta\right\rangle$

$= -\sum_{\substack{\alpha\in A\\ \beta\in B}}\lambda_\alpha\lambda_\beta\langle\alpha,\beta\rangle$

Claim If $\alpha\neq\beta$, $\alpha,\beta\in\Delta(\gamma)$

$\Rightarrow \langle\alpha,\beta\rangle \leq 0$.

(In fact if $\langle\alpha,\beta\rangle > 0$

Lemma $\Rightarrow \beta-\alpha$ and $\alpha-\beta\in\Sigma^+(\gamma)$.

We may assume $\beta-\alpha\in\Sigma^+(\gamma)$.

But then $\beta = (\beta-\alpha)+\alpha$, contradicting that β was indecomposable.)

$\Rightarrow \left\|\sum_{\alpha\in A}\lambda_\alpha\alpha\right\|^2 =$

$= -\sum_{\substack{\alpha\in A\\ \beta\in B}}\lambda_\alpha\lambda_\beta\langle\alpha,\beta\rangle \leq 0$

$\Rightarrow \sum_{\alpha\in A}\lambda_\alpha\alpha = 0$, $\lambda_\alpha\geq 0 \Rightarrow$

$\Rightarrow \lambda_\alpha \equiv 0$

$\Rightarrow \sum_{\beta\in B}\lambda_\beta\beta = 0$, $\lambda_\beta \leq 0 \Rightarrow$

$\Rightarrow \lambda_\beta \equiv 0$.

(3) Let $\Delta\subset\Sigma$ be any other basis. Since Δ is a basis of $\mathbb{E} \Rightarrow \exists\, \gamma\in\mathbb{E}$ s.t.

$\langle \alpha, \gamma \rangle > 0 \quad \forall \alpha \in \Delta$
(exercise existence of a positive quadrant, Helly's thm?)
$\Rightarrow \gamma \in \mathbb{E}$ is regular.
Moreover
$$\Sigma^+ \subseteq \Sigma^+(\gamma), \quad \Sigma^- \subseteq \Sigma^-(\gamma)$$
Since $\Sigma = \Sigma^+ \cup \Sigma^- = \Sigma^+(\gamma) \cup \Sigma^-(\gamma)$
$\Rightarrow \Sigma^+ = \Sigma^+(\gamma), \ \Sigma^- = \Sigma^-(\gamma)$
and $\Delta(\gamma)$ consists of indecomposable elements
$\Rightarrow \Delta(\gamma) \subset \Delta \Rightarrow \Delta(\gamma) = \Delta$ ☒

Rk If $\gamma, \gamma' \in$ the same Weyl chamber $\Rightarrow \Delta(\gamma) = \Delta(\gamma')$
sign $\langle \alpha, \gamma \rangle$ = sign $\langle \alpha, \gamma' \rangle$
$\Rightarrow \exists$ 1-1 corr. between bases and Weyl chambers.

If $\alpha \in \Sigma$, and σ_α = reflection w.r.t. P_α, define.
$$W := \langle \sigma_\alpha : \alpha \in \Sigma \rangle < O(\mathbb{E})$$
and is called the Weyl group of Σ.

If $w \in W \Rightarrow w(\Sigma) = \Sigma$, and $\forall w \in W, \alpha \in \Sigma$
$$w \sigma_\alpha w^{-1} = \sigma_{w(\alpha)}$$
$\Rightarrow W$ permutes the Weyl chambers and in fact, if γ is regular and $C(\gamma)$ = Weyl chamber that contains $\gamma \Rightarrow w(C(\gamma)) = C(w(\gamma))$.

Furthermore if Δ is a basis $\Rightarrow w(\Delta)$ is also a basis $\forall w \in W$. These actions are compatible, that is the map
$$\text{Weyl Chambers} \longrightarrow \text{Basis}$$
is W-equivariant.

Thm III.25 Let Δ be a basis of Σ.
(1) $W = \langle \sigma_\alpha : \alpha \in \Delta \rangle$
(2) W acts simply trans. on the set of basis
(3) W acts simply trans. on the set of Weyl chambers.

Lemma III.26 Let $\alpha \in \Delta \subset \Sigma$ reduced. Then σ_α permutes $\Sigma^+ \setminus \{\alpha\}$

Pf let $\beta \in \Sigma^+ \setminus \{\alpha\}$,
$$\beta = \sum_{\delta \in \Delta} c_\delta \delta, \quad c_\delta \in \mathbb{Z}_{\geq 0}$$
Since Σ is reduced $\Rightarrow$
$\Rightarrow \beta$ is not a multiple of α
$\Rightarrow c_\delta \neq 0$ for some $\delta \neq \alpha$.
But $\sigma_\alpha(\beta) = \beta - \frac{\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha$
$\Rightarrow$ the coeff. of δ in $\sigma_\alpha(\beta)$ is the same as the coeff. of δ in $\beta \Rightarrow \sigma_\alpha(\beta) \in \Sigma^+$. But also
$$\sigma_\alpha(\beta) \neq \alpha \Rightarrow \sigma_\alpha(\beta) \in \Sigma^+ \setminus \{\alpha\}.$$
☒

Lemma III.27 If $\beta = \sum_{\delta \in \Sigma^+} \delta$, then
$$\sigma_\alpha(\beta) = \beta - 2\alpha \quad \forall \alpha \in \Delta.$$
Pf $\sigma_\alpha(\beta) = \sigma_\alpha\left(\sum_{\delta \in \Sigma^+ \setminus \{\alpha\}} \delta + \alpha\right) \overset{III.26}{=}$

$$= \sum_{\delta \in \Sigma^+ \setminus \{\alpha\}} \delta + \underbrace{\sigma_\alpha(\alpha)}_{\substack{\shortparallel \\ -\alpha}} =$$

$$= \beta - 2\alpha .$$ $\blacksquare$