lecture

24 May 2021

Thm III.25 Δ basis of Σ

(1) $W = \langle \sigma_\alpha : \alpha \in \Delta \rangle$

(2) W acts simply trans. on bases

(3) " " Weyl ch.

Lemma III.27 If $\beta = \sum_{\delta \in \Sigma^+} \delta \Rightarrow$

$\Rightarrow \forall \alpha \in \Delta \quad \sigma_\alpha(\beta) = \beta - 2\alpha.$

Lemma III.28 $\alpha_1, \dots, \alpha_n \in \Delta$ (not nec. distinct) and write $\sigma_i := \sigma_{\alpha_i}$, $i = 1, \dots, n$. Assume $\sigma_1 \dots \sigma_n(\alpha_n) \in \Sigma^+ \Rightarrow \exists\, i$ $1 \le i \le n$ s.t. $\sigma_1 \dots \sigma_n = \sigma_1 \dots \hat{\sigma}_i \dots \sigma_{n-1}$.

Pf $\sigma_n(\alpha_n) = -\alpha_n \Rightarrow \sigma_1 \dots \sigma_{n-1}(\alpha_n) \in \Sigma^-$.

$\sigma_{n-1}(\alpha_n)$

$\vdots$

$\sigma_1 \dots \sigma_{n-2}\sigma_{n-1}(\alpha_n) \in \Sigma^-$

If $\sigma_{n-1}(\alpha_n) \in \Sigma^- \Rightarrow \alpha_n = \alpha_{n-1} \Rightarrow$

(because σ_α permutes $\Sigma^+ \setminus \{\alpha\}$ $\forall \alpha \in \Delta$)

$\Rightarrow \sigma_n = \sigma_{n-1} \Rightarrow \sigma_1 \dots \sigma_n = \sigma_1 \dots \sigma_{n-2}$

If $\sigma_{n-1}(\alpha_n) \in \Sigma^+ \Rightarrow$ let $1 \le i \le n-2$ be the smallest index s.t. $\sigma_{i+1} \dots \sigma_{n-1}(\alpha_n) \in \Sigma^+$ and $\sigma_i \sigma_{i+1} \dots \sigma_{n-1}(\alpha_n) \in \Sigma^-$.

Then $\underbrace{\sigma_{i+1} \dots \sigma_{n-1}}_{w}(\alpha_n) = \alpha_i$. Thus

$$\sigma_i = \sigma_{\alpha_i} = \sigma_{w(\alpha_n)} = w\sigma_{\alpha_n}w^{-1} \Rightarrow$$

$$\Rightarrow w\sigma_{\alpha_n} = \sigma_{\alpha_i} w \Rightarrow w\sigma_n = \sigma_i w$$

$$\Rightarrow \underbrace{\sigma_{i+1} \dots \sigma_{n-1}}_{w} \sigma_n = \sigma_i \underbrace{\sigma_{i+1} \dots \sigma_{n-1}}_{w}$$

$\Rightarrow$ multiply on the left by $\sigma_1 \dots \sigma_i \Rightarrow$

$$\Rightarrow \sigma_1 \dots \sigma_i \sigma_{i+1} \dots \sigma_{n-1}\sigma_n =$$

$$= \sigma_1 \dots \cancel{\sigma_i \sigma_i}\, \sigma_{i+1} \dots \sigma_{n-1} \quad \boxtimes$$

Lemma III.29 Let $\sigma = \sigma_{\alpha_1} \dots \sigma_{\alpha_n}$, where n is the minimal # of roots to write σ as a root w.r.t Δ. Then $\sigma(\alpha_n) \in \Sigma^-$.

Proof of Thm III.25

(1) Define $W' := \langle \sigma_\alpha : \alpha \in \Delta \rangle$ and show it acts trans. on Weyl chambers hence roots

(2) $W = W'$

(3) $W = W'$ ↻ freely on bases.

(1) We need the following:

Claim Let $\gamma \in E$ be regular. $\Rightarrow \exists\, \sigma \in W'$ s.t. $\langle \sigma(\gamma), \alpha \rangle > 0$ $\forall \alpha \in \Delta$.

Assuming this, if C is the Weyl chamber with $\gamma \in C \Rightarrow \sigma(C)$ is the Weyl chamber that contains $\sigma(\gamma)$. $\Rightarrow W$ ↻ trans. on Weyl chambers.

- To see why the claim is true, let $\delta = \sum_{\alpha \in \Sigma^+} \alpha$ and let $\sigma \in W'$ s.t. $\langle \sigma(\gamma), \delta \rangle$ is maximal.

$$\Rightarrow \forall \alpha \in \Delta \quad \langle \sigma(\gamma), \delta \rangle \ge$$

$$\ge \langle \sigma_\alpha(\sigma(\gamma)), \delta \rangle =$$

$$= \langle \sigma(\gamma), \sigma_\alpha(\delta) \rangle = \langle \sigma(\gamma), \delta - 2\alpha \rangle$$

$$= \langle \sigma(\gamma), \delta \rangle - 2\langle \sigma(\gamma), \alpha \rangle \Rightarrow$$

$$\Rightarrow \langle \sigma(\gamma), \alpha \rangle \ge 0 \quad \forall \alpha \in \Delta.$$

(2) We need the following:

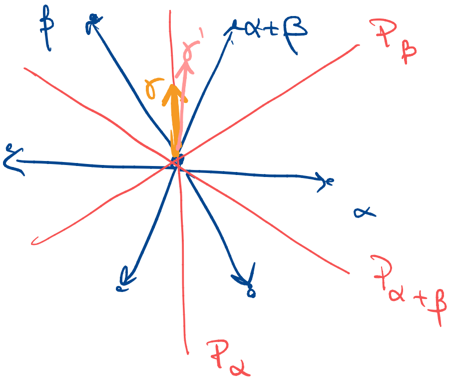
Claim If $\alpha \in \Sigma \Rightarrow \exists\, \sigma \in W'$ s.t. $\sigma(\alpha) \in \Delta$.

If so, given $\alpha \in \Sigma$ want to see that $\sigma_\alpha \in W'$. In fact

$$\sigma_\alpha = \sigma^{-1} \sigma_{\sigma(\alpha)} \sigma \in \underset{W'}{\sigma^{-1}} W' \underset{W'}{\sigma} = W'.$$

- To verify the claim, since W' ↻ $\{\text{bases}\}$ it is enough to show that any $\alpha \in \Sigma$ is in some basis.

To do this, let $\gamma \in P_\alpha \setminus \bigcup_{\beta \in \Sigma \setminus \{\pm\alpha\}} P_\beta$



Pick γ' close to γ s.t.
$\langle \gamma', \alpha \rangle = \varepsilon > 0$ and
$|\langle \gamma', \beta \rangle| > \varepsilon \quad \forall\ \beta \in \Sigma \setminus \{\pm\alpha\}$
$\Rightarrow \alpha \in \Sigma^+(\gamma)$ and is indecomposable.

(3) Let $\sigma \in W = W'$, $\sigma \neq e$ be such $\sigma(\Delta) = \Delta$. Let $\sigma = \sigma_{\alpha_1} \cdots \sigma_{\alpha_n}$ be the minimum product of simple reflections, $\alpha_j \in \Delta \subset \Sigma^+ \Rightarrow \sigma(\alpha_n) \in \Sigma^-$, which is a contradiction $\blacksquare$.

III.8 Iwasawa decomposition

Σ = root system associated to an OSLA $(\mathfrak{g}, \Theta)$ coming from a RSP (G, K).
$\mathfrak{g}_\alpha$ = root spaces of $\alpha \in \mathfrak{a}^* \setminus \{0\}$,
$\mathfrak{a}$ = max. Abelian subalgebra $\subset \mathfrak{p}$.
Fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$,
$\Sigma^+ = \{\alpha \in \Sigma : \alpha(H) > 0 \ \forall H \in \mathfrak{a}^+\}$.
Set $\mathfrak{n}^+ := \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$.

Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ & $\mathfrak{g}$ is f.d. $\Rightarrow \mathfrak{n}^+$ is nilpotent and $N^+ := \exp(\mathfrak{n}^+)$ a unipotent subgp of G.

Thm III.30 (Iwasawa decomp.)
We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$.
If $N^+ := \exp(\mathfrak{n}^+)$, $A = \exp \mathfrak{a}$

then the map
$$K \times A \times N^+ \longrightarrow G$$
$$(k, a, n) \longmapsto kan$$
is a diffeomorphism.

Sketch of the proof

[Helgason, Thm VI.3.4 p. 263
Thm VI.5.1 p. 270]

Let $X \in \sum_{\alpha \in \Sigma^-} \mathfrak{g}_\alpha$. We can write $X = \underbrace{(X + \Theta(X))}_{\in\, \mathfrak{k}} - \underbrace{\Theta(X)}_{\in\, \mathfrak{n}^+}$ $\quad \Theta\mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$

$\Rightarrow \sum_{\alpha \in \Sigma^-} \mathfrak{g}_\alpha \subset \mathfrak{k} + \mathfrak{n}^+$.

Moreover, since $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$
$\Rightarrow \mathfrak{g}_0 = \mathfrak{g}_0 \cap \mathfrak{g} = \mathfrak{g}_0 \cap \mathfrak{p} \oplus \mathfrak{g}_0 \cap \mathfrak{k}$
$= (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus \mathfrak{a}$

$$\Rightarrow \mathfrak{g} = \underset{(\mathfrak{g}_0 \cap \mathfrak{k}) \oplus \mathfrak{a}}{\underset{\parallel}{\mathfrak{g}_0}} + \underbrace{\sum_{\alpha \in \Sigma^-} \mathfrak{g}_\alpha}_{\subset\, \mathfrak{k} + \mathfrak{n}^+} + \underbrace{\sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha}_{= \mathfrak{n}^+}$$
$$\subset \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+. \quad \blacksquare$$

Let $M = G/K$ with basept $o \in M$.
$\Rightarrow M = N^+ . A . K . o = N^+ A . o$
is a "foliation of M by flats"
$A . o$ = flats.

Example $(SL(n,\mathbb{R}), SO(n,\mathbb{R}))$
$\mathfrak{a}^+ = \{ \operatorname{diag}(t_1, \dots, t_n),\ \sum t_j = 0,\ t_1 > \dots > t_n \}$
We saw that $\exists\ n(n-1)$ roots
$\alpha_{ij}(H) = t_i - t_j$ if $H = \operatorname{diag}(t_1 \dots t_n) \in \mathfrak{a}^+$
with corresponding root spaces
$\mathfrak{g}_{ij} = \mathbb{R} E_{ij}$. Then $\Sigma^+ = \{\alpha_{ij} : i < j\}$

and a basis is $\Delta = \{\alpha_i : i = 1, \dots, n-1\}$

Then $\mathfrak{n}^+ = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\} \subset \mathfrak{sl}(n, \mathbb{R})$

$$N^+ = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} < SL(n, \mathbb{R}).$$

$$A = \{ \operatorname{diag}(\lambda_1, \dots, \lambda_n) : \textstyle\prod \lambda_i = 1 \}$$

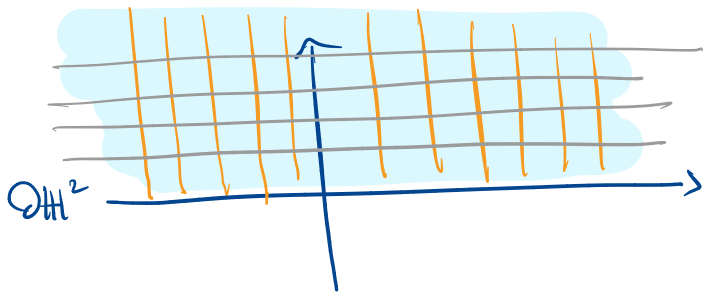
$\Rightarrow SL(n, \mathbb{R}) = SO(n) \cdot A \cdot N^+$
(= Gram-Schmidt)

To see the foliation by flats, take $n = 2$, so that $G/K = \mathbb{H}^2 \cong G \cdot i$

$$\underset{N^+}{\underset{\cap}{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}}} \underset{A}{\underset{\cap}{\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}}} \cdot i = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} e^t i = e^t \cdot i + x,$$

$x, t \in \mathbb{R}$.

For a fixed x, the set
$e^t \cdot i + x$, $t \in \mathbb{R}$ is a geodesic



For a fixed t, the set
$e^t \cdot i + x$, $x \in \mathbb{R}$ is a horocycle
and is an N^+-orbit.