lecture

2 June 2021

III.8 More about the Weyl group

(G,K) effective RSP of non-cpt type, $\mathfrak{a}\subset\mathfrak{p}$ max. Ab. and $\Sigma\subset\mathfrak{a}^*$ roots.

$W(\Sigma) < O(\mathfrak{a}, \langle\,,\rangle)$ generated by refl. in Σ or Δ = basis.

This holds $\forall$ abstract root system.

$N_K(\mathfrak{a}) := \{k\in K : Ad_G(k)(\mathfrak{a}) = \mathfrak{a}\}$

$Centr_K(\mathfrak{a}) := \{k\in K : Ad_G(k)|_{\mathfrak{a}} = id_{\mathfrak{a}}\}$

Thm III.31 The restriction map

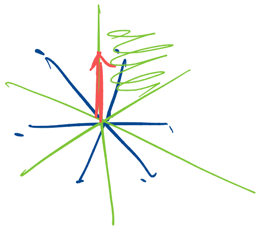
$$N_K(\mathfrak{a}) \longrightarrow O(\mathfrak{a}), \qquad k \longmapsto Ad_G(k)$$

induces an isom.

$$N_K(\mathfrak{a})/Centr_K(\mathfrak{a}) \cong W(\Sigma).$$

No proof: Im $N_K(\mathfrak{a}) \supset W(\Sigma)$ easy, the issue is to show that it does not contain more than this.

Corollary III.32 Let $F := (\exp\mathfrak{a})\cdot o$ and $Stab_G(F) := \{g\in G : gF = F\}$.
Then the image of $Stab_G(F)$ under $Stab_G(F) \longrightarrow Iso(F)$ is isom. to the semidirect product $W(\Sigma)\ltimes\mathfrak{a}$.



Proof (•) $g\in Stab_G(F)$ ⇝ write it as $g = ak \in A\cdot N_K(\mathfrak{a})$
$a = \exp(Y)$, $Y\in\mathfrak{a}$

(•) write explicitly the G-action on F as a direct product.

Since $g\in Stab_G(F)$, $F = \exp(\mathfrak{a})\cdot o$

$\Rightarrow g(\exp(\mathfrak{a})\cdot o) = \exp(\mathfrak{a})\cdot o$.

Let $a\in\mathfrak{a}$ be such that
$g\cdot o = a\cdot o$

$\Rightarrow k = a^{-1}g\in K$. We claim that $k\in N_K(\mathfrak{a})$. In fact

$$\exp(\mathfrak{a})\cdot o = k(\exp\mathfrak{a}\cdot o) =$$

$$= \underbrace{k\exp\mathfrak{a}\,k^{-1}}_{\shortparallel \; Ad_G(k)}\cdot o \quad \Rightarrow$$

$\Rightarrow \exp(\mathfrak{a})\cdot o = \exp(Ad_G(k)\mathfrak{a})\cdot o$

$\Rightarrow \mathfrak{a} = Ad_G(k)\mathfrak{a} \Rightarrow$

$\Rightarrow k\in N_K(\mathfrak{a})$.

Now $g = ka$, $k\in N_K(\mathfrak{a})$, $a\in\mathfrak{a}$.

$$g(\underbrace{\exp X\cdot o}_{\in F}) = ka\exp X\cdot o) =$$

$X\in\mathfrak{a}$

$Y\in\mathfrak{a}$ with $\exp(Y) = a$

$$= \overbrace{\exp(Y)}^{a}k\exp X\cdot o =$$

$$= \exp(Y)k\exp X\,k^{-1}\cdot o$$

$$= \exp(\underset{\in\,\mathfrak{a}}{Y})\exp(\underset{\in\,\mathfrak{a}}{Ad(k)X})\cdot o$$

$$= \exp(\underset{\in\,\mathfrak{a}}{Y} + \underset{\in\,W(\Sigma)}{Ad(k)}X)\cdot o$$

by Thm III.31 ☒

Also needed to prove Thm III.31

Thm III.33 Let $X\in\mathfrak{p}$ and $\mathcal{C}(X)$ the Weyl chamber ⇒

$\Rightarrow \forall k\in K$ s.t. $Ad_G(k)\mathcal{C}(X) = \mathcal{C}(X)$

then $Ad(k)Y = Y \quad \forall Y\in E_X$?

$(\mathfrak{g},\Theta)$ OSLA associated to RSP (G,K). If $X\in\mathfrak{p}$ let

$$E_X := \bigcap_{X\in\mathfrak{a}} \{\mathfrak{a}\subset\mathfrak{p} \text{ max. Ab. st.}\}$$

= lin space of the inters. of all flats

containing X. (One can also write $E_X = \bigcap_{\substack{\alpha\in\Sigma \\ \alpha(X)\neq 0}} \ker\alpha$.)

Proven: X regular $\iff$ it belongs to a ! max-flat

$\Rightarrow$ X regular $\iff$ E_X Max. Ab.

$(E_X)_r := \{ Z \in E_X : E_Z = E_X \}$.

Let $\mathcal{C}(X)$ be the connected component of $(E_X)_r$ that contains X. Then

$$\mathcal{C}(X) = \begin{cases} \text{Weyl chamber} & X \text{ regular} \\ \text{Weyl face} & X \text{ not reg.} \end{cases}$$

Ex $SL(3,\mathbb{R})/SO(3,\mathbb{R})$

$$\mathfrak{a} = \left\{ \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} : x_1+x_2+x_3=0 \right\}$$

$N_K(\mathfrak{a})$ = monomial matrices, that is matrices whose columns are $\pm e_i, \pm e_j, \pm e_k$ with exactly one 1 in each row & each column and $\det = 1$.

$$\mathrm{Centr}_K(\mathfrak{a}) = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix} \right\}.$$

$N_K(\mathfrak{a})/\mathrm{Centr}_K(\mathfrak{a}) \cong S_3$

$$\sigma \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_{\sigma(1)} \\ x_{\sigma(2)} \\ x_{\sigma(3)} \end{pmatrix}$$

$$\mathfrak{a}^+ = \left\{ \begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix} : \begin{matrix} x_1+x_2+x_3=0 \\ x_1 > x_2 > x_3 \end{matrix} \right\}$$

f.d. for the Weyl gp action on $\mathfrak{a}$

$$\overline{\mathfrak{a}^+} = \left\{ \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} : \begin{matrix} x_1+x_2+x_3=0 \\ x_1 \geq x_2 \geq x_3 \end{matrix} \right\}$$

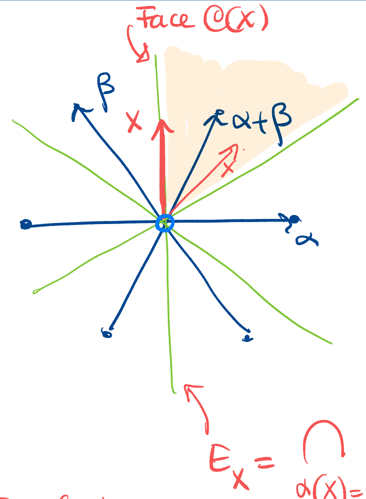
Faces

$$X_1 \in F_1 = \left\{ \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} : x_1 = x_2 \geq x_3 \right\}$$

$$X_2 \in F_2 = \left\{ \begin{pmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{pmatrix} : x_1 \geq x_2 = x_3 \right\}$$

$$X_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \quad \text{for example}$$

$$X_2 = \begin{pmatrix} 2 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$



$E_X = \bigcap_{\alpha(X)=0} \ker\alpha$

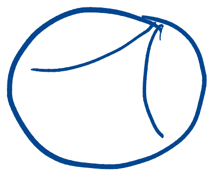
$(E_X)_r = E_X \setminus \{0\}$ ▨

III.8 Parabolic subgroups

Defn γ, η geodesics are asymptotic if $\exists\, C \geq 0$ s.t. $d(\eta(t), \gamma(t)) \leq C$ $\forall t \geq 0$. wr equivalence relation.

M (CAT(0)) RSS of non-cpt type $\Rightarrow$ $M(\infty) = \{ [\gamma] : \gamma \text{ geodesic} \}$

= visual boundary



$M(\infty) \cong (n-1)$ sphere S^{n-1}

$\cong T^1_p M$ = unit vectors at p.

$\bar{M} = M \cup M(\infty)$ with the cone topology is compact and homeo to a closed unit ball.

Let (G,K) be a reduced RSP of non-cpt type

Defn A parabolic subgp of G is $G_\xi := \{g \in G : g\xi = \xi\}$, for $\xi \in M(\infty)$.

Rk $o \in M \Rightarrow \mathrm{Stab}_G(o)$ cpt

$\xi \in M(\infty) \Rightarrow \mathrm{Stab}_G(\xi)$ not cpt.

Also G_ξ not connected, but if $(G_\xi)^\circ = (G_\eta)^\circ \Rightarrow G_\xi = G_\eta$.

Proposition III.34 Let $\xi \in M(\infty)$.

(1) $G\cdot\xi = K\cdot\xi$

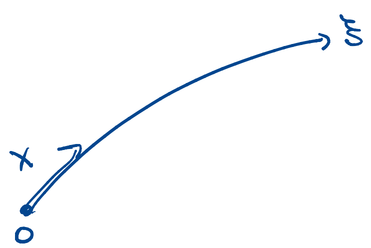
(2) $(G_\xi)^\circ$ acts trans. on M.

Pf (1) Let $X \in \mathfrak{p}$ s.t.

$\gamma_{o,\xi}(t) := \exp(tX).o$

$\gamma_{o,\xi}(0) = o$, $\gamma'_{o,\xi}(0) = X$, $\gamma_{o,\xi}(\infty) = \xi$

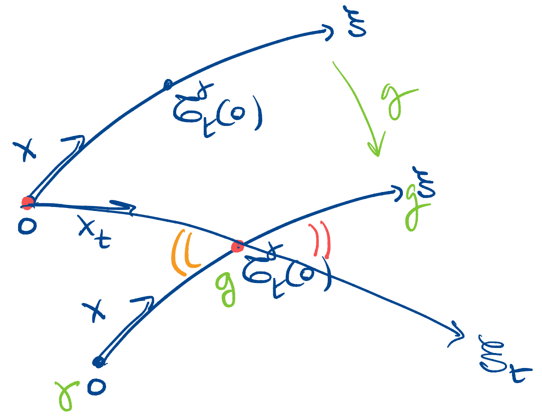
$\gamma_{o,\xi}$ represents ξ.



$\tau_t^{\gamma_{o,\xi}}(t) =$

$= \exp(tX) \in G$

$\tau_t^{\gamma_{o,\xi}} = \tau_t^{\gamma}$



Let $\eta_{o,\xi_t}(s) := \exp(sX_t).o$ be the geod. between o and $g\tau_t^\gamma(o)$ and τ_s^η the corresponding 1-par. gp. of transvections.

Let $s_t \in [0,\infty)$ be s.t.

$\tau^\eta_{s_t}(o) = g\,\tau^\gamma_t(o) \Rightarrow$

$k_t := \tau^\eta_{s_t} \circ g \circ \tau^\gamma_t$

$\measuredangle_o(k_t\xi, \xi_t) =$

$= \measuredangle_{\tau^\eta_{s_t} o}\big(\underbrace{\tau^\eta_{s_t}\cdot k_t\xi}_{= g\tau_t^\gamma \xi \,=\, g\xi}, \underbrace{\tau^\eta_{s_t}\cdot \xi_t}_{= \xi_t}\big)$

$= \measuredangle_{\tau^\eta_{s_t} o}(g\xi, \xi_t) =$)) $=$

$=$ (($= \measuredangle_{g\tau_t^\gamma(o)}(o, \gamma o)$

As $t \to \infty$ $\xi_t \to g\xi \Rightarrow$

$\Rightarrow \measuredangle_o(k_t\xi, \xi_t) \longrightarrow 0$

$\Rightarrow \lim_{t\to\infty} k_t\xi = g\xi$.

Take a conv. subsequence and show that $k\xi = g\xi$.

(2) $p,q \in M$ and let $g \in G$ be s.t. $gp = q$. By (1) if $\xi \in M(\infty)$ $\exists\, k \in K$ s.t. $k\xi = g^{-1}\xi$. Set $h = gk$ $\Rightarrow$

$\Rightarrow h\xi = gk\xi = \xi \Rightarrow h \in G_\xi$

and $h(p) = gk(p) = q$.

$\Rightarrow G_\xi \curvearrowright M$ transit.

$\Rightarrow (G_\xi)^\circ \curvearrowright M$ trans. (since M is connected). $\square$

Let $X \in \mathfrak{p}$ repr. $\xi \in M(\infty)$.

$$T_\xi : G_\xi \longrightarrow G$$
$$g \longmapsto \lim_{t\to\infty} \exp(-tX)\, g \exp(tX)$$

Remark

(1) $g \in G_\xi \iff \exists \lim_{t\to\infty} \exp(-tX)\, g \exp(tX)$

($\Leftarrow$) $d(\gamma_{o,\xi}(t), g\gamma_{o,\xi}(t)) =$

$= d(\exp(tX)\cdot o,\ g \exp(tX)\cdot o)$

$= d(o,\ \underbrace{\exp(-tX)\, g \exp(tX)}\cdot o)$

$\longrightarrow$ finite limit. $\Rightarrow$

$\Rightarrow g \in G_\xi$.

($\Rightarrow$) More convoluted

(2) $\mathrm{Im}(T_\xi) \subset G_\xi$.

Notation $\xi \in M(\infty)$ repr. by X

$Z_\xi := \{ g \in G : \mathrm{Ad}(g) X = X \}$

$K_\xi := Z_\xi \cap K$

$N_\xi := \ker T_\xi$

$A_\xi := \exp(Z(X) \cap \mathfrak{p})$, $Z(X) = \{ Y \in \mathfrak{g} : [X,Y] = 0 \}$

Proposition III.35

(1) N_ξ is a connected normal subgp of G_ξ. (conn. since we have a path to e)

(2) $\mathrm{Im}(T_\xi) = Z_\xi$ (after we contracted to what we could, the rest commutes with X).

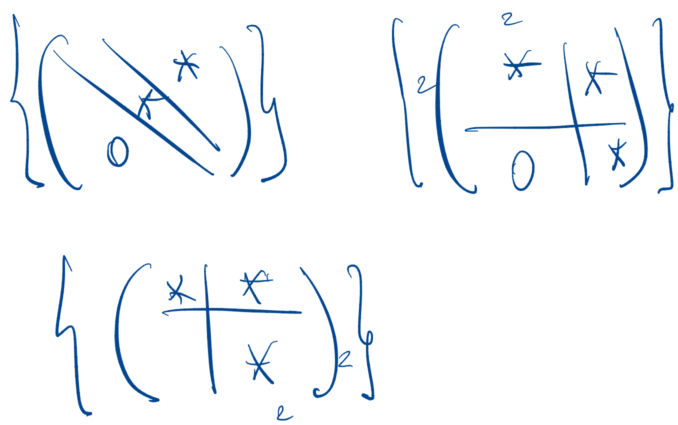
(3) $G_\xi = K_\xi A_\xi N_\xi$

(in general $Z(X) \cap \mathfrak{p}$ is only a vector subspace. However if X is regular $\Rightarrow$ $Z(X) \cap \mathfrak{p}$ is Maximal Abelian and hence $G_\xi = K_\xi A_\xi N_\xi$ is the Iwasawa decomp. of G_ξ.)

Lemma III.36 $\mathfrak{a} \subset \mathfrak{p}$ max Ab. $\Sigma \subset \mathfrak{a}^*$ roots. Then

$$\mathfrak{g}_\xi = \mathfrak{g}_0 \oplus \sum_{\alpha(X) \geq 0} \mathfrak{g}_\alpha$$

(here X represents ξ)



$\exists!$ conj. class of K

$\exists$ f.m. conj. classes of parabolic sbgp.