lecture

3 June 2021

$\xi \in M(\infty)$ repr. by $X \in \mathfrak{p}$.

$G_\xi = \{g \in G : g\xi = \xi\} =:$ parabolic subgp.

(1) $G \cdot \xi = K \cdot \xi \quad \forall\, \xi \in M(\infty)$

(2) $(G_\xi)^\circ$ acts trans. on M.

$\tau_\xi : G_\xi \longrightarrow G$

$g \longmapsto \lim_{t\to\infty} \exp(-tX)\, g \exp(tX)$

$\Rightarrow G_\xi \overset{(*)}{=} K_\xi A_\xi N_\xi$,

$N_\xi = \ker \tau_\xi$

$A_\xi = \exp(Z(X) \cap \mathfrak{p})$,

$Z(X) = \{Y \in \mathfrak{g} : [X,Y] = 0\}$

$K_\xi = K \cap \{g \in G : \mathrm{Ad}(g)X = X\}$

If X regular $\Rightarrow Z(X) \cap \mathfrak{p}$ max. Ab $\Rightarrow (*)$ is the Iwasawa dec. of G_ξ.

If $\mathfrak{a} \subset \mathfrak{p}$ is a fixed max. Ab.

with $\Sigma \subset \mathfrak{a}^* \setminus \{0\} \Rightarrow$

$\mathfrak{g}_\xi \overset{(*)}{=} \mathfrak{g}_0 \oplus \sum_{\alpha(X)\geq 0}^{\oplus} \mathfrak{g}_\alpha$

Rk $\rightsquigarrow$ ordering on parabolics, useful to study various boundaries.

Corollary III.37 Let $X, Y \in \mathfrak{p}$ with $\mathcal{C}(X) = \mathcal{C}(Y)$. Let $\xi = \exp(\infty \cdot X)$, $\eta = \exp(\infty Y) \Rightarrow G_\xi = G_\eta$

Why is this true By defn, since $\mathcal{C}(X) = \mathcal{C}(Y) \Rightarrow$ the roots ≥ 0 on X are exactly the roots ≥ 0 on Y. The assertion then follows from $(*)$.

Want to define $\xi \in M(\infty)$ "regular"

$\underbrace{r_{p,X}(\infty)}_{\text{geod. ray}} = \xi$. X regular $\Rightarrow r_{p,X}$ is cont. in ω! max. flat. Let $q \in M$ and $Y \in \mathfrak{p}$ s.t. $r_{q,Y}(\infty) = \xi$

$\Rightarrow \exists\, g \in G_\xi$ s.t. $g\, r_{p,X} = r_{q,Y}$.

That is $q = g \cdot p$, $d_p g\, X = Y$,

$\Rightarrow r_{q,Y}$ is also contained in $\mathfrak{a}$! max. flat.

Defn. $\xi \in M(\infty)$ is regular if a (hence any) geodesic repr. ξ is contained into a max. flat.

Rk ξ regular $\Leftrightarrow X$ regular and the set of reg. pts in $M(\infty)$ is a dense open subset in $M(\infty)$ that is G-invariant.

Let $\xi \in M(\infty)$, $p \in M$, $X \in \mathfrak{p}$ repr. ξ.

$\mathcal{C}_p(\xi) := \{\exp(\infty Y) \cdot p : Y \in \mathcal{C}(X)\}$

Lemma III.38 $\mathcal{C}_p(\xi) = \mathcal{C}_q(\xi)$ $\forall\, p, q \in M$.

(Uses again the $(G_\xi)^\circ \curvearrowright M$ trans.)

Defn. If $\xi \in M(\infty)$ is regular $\Rightarrow \mathcal{C}(\xi) := \mathcal{C}_p(\xi)$ $\forall\, p \in M$.

Rk (1) We'll define $\mathcal{C}(\xi)$ for non regular ξ using G_ξ.

(2) If ξ, η are regular and $\mathcal{C}(\xi) \cap \mathcal{C}(\eta) \neq \emptyset \Rightarrow$

$\Rightarrow \mathcal{C}(\xi) = \mathcal{C}(\eta)$.

Thm III.39 If $\xi, \eta \in M(\infty)$ $\Rightarrow \exists$ a maximal flat F s.t. $\xi, \eta \in F(\infty)$.

Rk Important because $\rightsquigarrow$ second axiom of building.

Next Relation between the tessell. of $M(\infty)$ by Weyl Ch. and the family of parabolic subgps.

Proposition III.40

Let $p \in M$ and let $\xi, \eta \in M(\infty)$ represented resp. by $X, Y \in \mathfrak{F}$. Then:

(1) $\mathcal{C}(X) = \mathcal{C}(Y) \iff (G_\xi)^\circ = (G_\eta)^\circ$

(2) $(G_\xi)^\circ = (G_\eta)^\circ \iff G_\xi = G_\eta$

(3) $\mathcal{C}(X) \supseteq \mathcal{C}(Y) \iff$

$(G_\xi)^\circ \subseteq (G_\eta)^\circ \overset{(2)}{\iff} G_\xi \subseteq G_\eta$

(4) $N_G(G_\xi) = G_\xi$.

Why this is True

(1) ($\Rightarrow$) done

($\Leftarrow$) Thm III.39 $\exists F$ s.t. $\xi, \eta \in F(\infty)$. $F = \exp(\mathfrak{a}) \cdot p$

Since $(G_\xi)^\circ = (G_\eta)^\circ$ we may assume that $p = 0$ $\rightsquigarrow$ we translate the picture in $\mathfrak{F}$.

We use then that

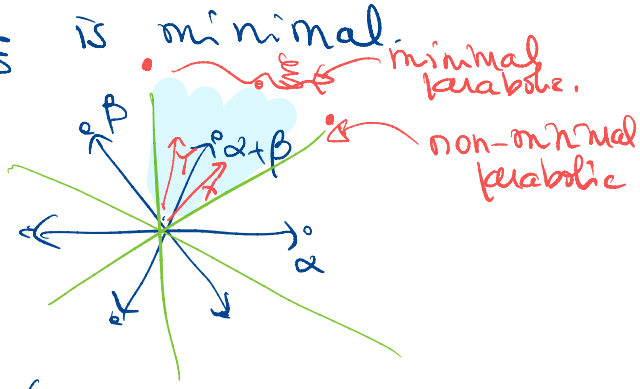
$$\mathfrak{g}_\xi = \mathfrak{g}_0 \oplus \sum_{\alpha(X) \geq 0}^{\oplus} \mathfrak{g}_\alpha$$

to conclude.

Defn. A parabolic subgp G_ξ is minimal if whenever $(G_\xi)^\circ \supseteq (G_\eta)^\circ \Rightarrow (G_\xi)^\circ = (G_\eta)^\circ$

Proposition III.41 A point $\xi \in M(\infty)$ is regular iff G_ξ is minimal.

Ex.



$\left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ min. par.

$\left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \right\}$

$\left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \right\}$

non-minimal par. that correspond to "faces"

Defn. If $\xi \in M(\infty) \Rightarrow$

$\mathcal{C}(\xi) := \{ \eta \in M(\infty) : (G_\xi)^\circ = (G_\eta)^\circ \}$

Rk ξ regular $\Rightarrow$ Weyl ch.

ξ not regular $\Rightarrow$ Weyl face.

We can consider the visual boundary of a Weyl chamber or Weyl face $\mathcal{C}(X)$

$\mathcal{C}(X)(\infty) = \{ \exp(\infty Y) : Y \in \mathcal{C}(X) \}$

Corollary III.41

(1) If $\mathcal{C}(X) = \mathcal{C}(Y) \Rightarrow$

$\Rightarrow \mathcal{C}(X)(\infty) = \mathcal{C}(Y)(\infty)$

(2) The tessellation of $M(\infty)$ by $\{ \mathcal{C}(\xi) : \xi \in M(\infty) \}$ coincides with the

tessellation $\{\mathcal{C}(X)(\infty) : X \in \mathcal{F}\}$ by Weyl chambers and Weyl faces.

Thm III.42 Let $\mathcal{F}$ max. flat and $\xi \in \mathcal{F}(\infty)$ be a regular point. If $\eta \in M(\infty)$ is any pt, the G_ξ-orbit $G_\xi \cdot \eta$ intersects $\mathcal{F}(\infty)$ in exactly one pt.

Bruhat decomposition

Let $\xi \in \mathcal{F}(\infty)$ be a regular pt and if $g \in G$ let $\eta := g\xi$. Then η is regular as well).

By Thm III.42 $\Rightarrow \exists\, h \in G_\xi$ s.t. $hg\xi \in \mathcal{F}(\infty)$. $\Rightarrow \exists\, w \in W$ s.t.

$$\underbrace{w\,\mathcal{C}(hg\xi)}_{\mathcal{C}(whg\xi)} = \mathcal{C}(\xi) \xRightarrow{\text{Prop. III.40 (1)(2)}}$$

$$\Rightarrow G_{whg\xi} = G_\xi$$

$$G_{whg\xi} = (whg)^{-1} G_\xi (whg) \Rightarrow$$

$\Rightarrow whg$ normalizes G_ξ

Prop. III.40 (4) $\Rightarrow whg \in G_\xi$

$\Rightarrow whg = h' \in G_\xi \Rightarrow$

$\Rightarrow g = h^{-1} w^{-1} h' \Rightarrow G = \bigcup_{w \in W} G_\xi w G_\xi$

which is the Bruhat decomposition of G.

How to get $\mathcal{B}$

$\mathcal{F}(\infty)$ has the structure of Coxeter complex: decompose it into simplicial cones (i.e. Weyl chambers & Weyl faces) $\rightsquigarrow$ decompose $\mathcal{F}(\infty)$.

Another way to see this is to consider the $\cap$ of these simplicial cones with the unit sphere in $\mathfrak{a}$.

Defn. A spherical building $\mathcal{B}$ is a simplicial complex $\mathcal{B}$ together with a family $\mathcal{A}$ of subcomplexes, called apartments whose union is $\mathcal{B}$ and such that

(1) Each apartment is a Coxeter complex

(2) For any two simplices $\exists$ an apartment that contains them

(3) If $A, A' \in \mathcal{A}$ with a common chamber $\Rightarrow \exists$ isom. $A \to A'$ that fixes the simplex in $A \cap A'$.

Rk (1) $\mathcal{A}$ is not part of data

(2) In general the isom. in (3) does not come from an autom. of $\mathcal{B}$.

In our context a chamber is a simplex of max. dim: if the Coxeter cplx. comes

from a sym. space of rank $r \Rightarrow$ chambers are r-simplices.

Vertices of $\mathcal{B}$ are the pts at ∞ of the Weyl cones of rank 1.

$$\left[\begin{array}{l} E_X = \bigcap \{\mathfrak{a} \text{ max. Ab} : X \in \mathfrak{a}\} \\ \Rightarrow \exp(E_X) = \bigcap_{\substack{p \in F \\ \xi \in F(\infty)}} F =: E_p(\xi) \\ \text{rk } \mathcal{C}(\xi) = \dim E_p(\xi) \end{array}\right]$$

$$V = \{\xi \in M(\infty) : \underbrace{\text{rk } \mathcal{C}(\xi) = 1}_{\mathcal{C}(\xi) = \{\xi\}}\}$$

To every $\mathcal{C}(\xi)$ we associate

$$\partial(\mathcal{C}(\xi)) := \{\eta \in V : \eta \in \overline{\mathcal{C}(\xi)}\}$$

$$\Rightarrow \mathcal{B} := \{\partial(\mathcal{C}(\xi)) : \xi \in M(\infty)\}$$

is the proposed building with apartments.

$$A_F := \{\partial(\mathcal{C}(\xi)) : \xi \in F(\infty)\}.$$

where F is a max. flat.

$$\Rightarrow \mathcal{B} = \bigcup_{\substack{F \subset M \\ \text{max. flat}}} A_F$$

The axiom (2) follows from Thm III.39.

To see that $\mathcal{B}$ satisfies axiom (3) use the uniqueness in the Bruhat decom. (Thm III.42). F max. flat, $\xi \in F(\infty)$ regular

$$p : M(\infty) \longrightarrow F(\infty)$$
$$\eta \longmapsto G_\xi \eta \cap F(\infty)$$

$p|_{F(\infty)} = id_{F(\infty)}$ and p maps chambers to chambers $g\mathcal{C}(\eta) = \mathcal{C}(g\eta) \subset F(\infty)$

Observe that G is transitive on

$$\{(F, \mathcal{C}(\xi)) : F \text{ max flat}, \mathcal{C}(\xi) \subset F\}$$

Let F, F' be max. flats s.t. $F(\infty) \cap F'(\infty) \supset \mathcal{C}(\xi)$

$\Rightarrow \exists\, g \in G$ s.t. $gF' = F$ and preserves $\mathcal{C}(\xi) \Rightarrow$

$\Rightarrow g|_{F'(\infty)} : F'(\infty) \longrightarrow F(\infty)$ is an isom. that fixes $\mathcal{C}(\xi) =$ simplex in $F(\infty) \cap F'(\infty)$.

Application

"Mostow rigidity thm"

Let $\Gamma\backslash M$, $\Gamma'\backslash M'$ be two compact irreducible locally sym. spaces of non-cpt type. Assume $M, M' \neq \mathbb{H}^2$.

If $\Gamma\backslash M \simeq \Gamma'\backslash M'$ are homotopic equivalent ($\Gamma \simeq \Gamma'$) $\Rightarrow$

$\Rightarrow$ $\Gamma\backslash M$ & $\Gamma'\backslash M'$ are "isometric". (up to renormalizz.)

Why? $rk \geq 2$

$\Gamma\backslash M$, $\Gamma'\backslash M'$ homotopic $\Rightarrow$

$\Rightarrow$ $\varphi : \Gamma\backslash M \longrightarrow \Gamma'\backslash M'$ hom. equivalence $\rightsquigarrow$ quasi-isom.

$\Gamma\backslash M \to \Gamma'\backslash M'$

Using $\mathcal{B}$, $\mathcal{B}'$ $\Rightarrow$ the q.i. extends to an isom.

$$\mathcal{B} \longrightarrow \mathcal{B}'$$

$\rightsquigarrow$ the locally sym. spaces are isometric.

Let F, F' be max. flats s.t. $F(\infty) \cap F'(\infty) \supset \mathcal{C}(\xi)$

$\Rightarrow \exists\, g \in G$ s.t. $gF' = F$ and preserves $\mathcal{C}(\xi)$ $\Rightarrow$

$\Rightarrow g|_{F'(\infty)} : F'(\infty) \longrightarrow F(\infty)$

B an isom. that fixes $\mathcal{C}(\xi) =$ simplex in $F(\infty) \cap F'(\infty)$.

Application

Mostow rigidity thm"

Let $\Gamma\backslash M$, $\Gamma'\backslash M'$ be two compact irreducible locally symm. spaces of non-cpt type. Assume $M, M' \neq \mathbb{H}^2$.

If $\Gamma\backslash M \simeq \Gamma'\backslash M'$ are homotopic equivalent ($\Gamma \simeq \Gamma'$) $\Rightarrow$