NOTES ON DOMAINS OF TYPE A

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Definition 1.1. Let $\Omega \subset \mathbb{R}^n$ be open. We say that Ω is **of type *A***, for some positive constant $A > 0$, if for every $x \in \Omega$ and for every $0 < r < \min\{1, \mathrm{diam}(\Omega)\} =: r_0(\Omega)$ it holds that

$$\frac{\mathcal{L}^n\big(\Omega \cap B_r(x)\big)}{r^n} \geqslant A.$$

Remark 1.1. Notice that for every every $\mathcal{L}^n$-measurable set $E \subset \mathbb{R}^n$, it holds that

$$\frac{\mathcal{L}^n\big(E \cap B_r(x)\big)}{r^n} \leqslant \frac{\mathcal{L}^n\big(B_r(x)\big)}{r^n} = \omega_n,$$

where $\omega_n := \mathcal{L}^n\big(B_1(0)\big)$. In particular, an open set $\Omega \subset \mathbb{R}^n$ is of type A, for some $A > 0$, if and only if for every $x \in \Omega$ and for every $0 < r < r_0(\Omega)$ if holds that

$$A \leqslant \frac{\mathcal{L}^n\big(\Omega \cap B_r(x)\big)}{r^n} \leqslant \omega_n. \tag{1.1}$$

Equation (1.1) has a deep geometric meaning: if Ω is of type A, for every fixed point $x \in \Omega$ the measure of the part of the ball $B_r(x)$ centred at x which is contained in Ω cannot become too little nor too big, i.e. it cannot increase above $\omega_n r^n$ nor decrease below Ar^n, for any given $0 < r < r_0(\Omega)$.

Proposition 1.1. *Let $\Omega \subset\subset \mathbb{R}^n$ be open of class C^1. Then, Ω is of type A for some $A > 0$.*

Proof. Indeed, by contradiction, assume that we can find two sequences $\{x_k\}_{k\in\mathbb{N}} \subset \Omega$ and $\{r_k\}_{k\in\mathbb{N}} \subset \big(0, r_0(\Omega)\big)$ such that

$$\frac{\mathcal{L}^n\big(\Omega \cap B_{r_k}(x_k)\big)}{r_k^n} \to 0^+, \tag{1.2}$$

as $k \to +\infty$.
First, we claim that

$$\inf_{k\in\mathbb{N}} \mathrm{dist}(x_k, \partial\Omega) = 0. \tag{1.3}$$

If not, then we set

$$\tilde{r} := \inf_{k\in\mathbb{N}} \mathrm{dist}(x_k, \partial\Omega) > 0$$

and we notice that

$$\frac{\mathcal{L}^n\big(\Omega \cap B_{r_k}(x_k)\big)}{r_k^n} \geqslant \min\left\{\omega_n, \frac{\mathcal{L}^n\big(\Omega \cap B_{\tilde{r}}(x_k)\big)}{r_0(\Omega)^n}\right\} > 0,$$

which would contradict (1.2). Thus, since we have established (1.3) and since the sequence $\{x_k\}_{k\in\mathbb{N}}$ is bounded in $\mathbb{R}^n$, by the properties of the infimum and by Bolzano-Weierstrass theorem we can take a subsequence (not relabeled) such that $\mathrm{dist}(x_k, \partial\Omega) \to 0^+$ and $x_k \to x_0 \in \mathbb{R}^n$. By continuity of the function $\mathrm{dist}(\cdot, \partial\Omega)$, we get $\mathrm{dist}(x_0, \partial\Omega) = 0$. Since $\partial\Omega$ is a closed set, we conclude $x_0 \in \partial\Omega$.
Since Ω is of class C^1, we can find an open neighbourhood $U \subset \mathbb{R}^n$ of x_0 in $\mathbb{R}^n$ and a C^1-diffeomorphism $\varphi : U \to V := \varphi(U)$ such that $\varphi(x_0) = 0$, $\varphi(U \cap \Omega) = V \cap \{(x_1, ..., x_n) \text{ s.t. } x_n > 0\} =: V^+$ and $\varphi(U \cap \partial\Omega) = V \cap \{(x_1, ..., x_n) \text{ s.t. } x_n = 0\}$.

Up to reducing the size of U (and V consequently), we can assume that φ is C^1 up to the boundary of U, so that both φ and φ^{-1} are Lipschitz on U. We set

$$\begin{aligned} M &:= \max_U\{J\varphi\} = \max_U |\det(\nabla\varphi)| < +\infty, \\ L &:= \mathrm{Lip}(\varphi^{-1}) < \infty, \end{aligned}$$

where $\nabla\varphi$ denotes the Jacobian matrix of φ and $\mathrm{Lip}(\varphi^{-1})$ stands for the Lipschitz constant of the function φ^{-1} on V. Moreover, since $x_k \to x_0$ and $r_k \to 0^+$, by possibily discarding a finite number of elements in the sequence we can assume that $B_{r_k}(x_k) \subset U$, for every $k \in \mathbb{N}$.
Now notice that, by a simple change of variables, we obtain

$$\begin{aligned} \frac{\mathcal{L}^n\big(\Omega \cap B_{r_k}(x_k)\big)}{r_k^n} &= \frac{1}{r_k^n}\int_{\Omega\cap B_{r_k}(x_k)} d\mathcal{L}^n = \frac{1}{r_k^n}\int_{\varphi(\Omega\cap B_{r_k}(x_k))} (J\varphi)^{-1}\, d\mathcal{L}^n \\ &\geqslant M^{-1}\frac{\mathcal{L}^n\big(V^+ \cap \varphi(B_{r_k}(x_k)\big)}{r_k^n} \geqslant M^{-1}\frac{\mathcal{L}^n\big(V^+ \cap B_{Lr_k}(\varphi(x_k))\big)}{r_k^n}, \end{aligned} \tag{1.4}$$

where the second inequality comes from the fact that, since φ^{-1} is Lipschitz on U with Lipschitz constant $L > 0$, it follows that $\varphi\big(B_{r_k}(x_k)\big) \supset B_{Lr_k}(\varphi(x_k))$. It's very easy to see that for every $y \in V^+$ and $r > 0$ such that $B_r(y) \subset V$ it holds that

$$\frac{\mathcal{L}^n\big(V^+ \cap B_r(y)\big)}{r^n} \geqslant \frac{\omega_n}{2}.$$

Since by construction $\varphi(x_k) \in V^+$ and $\varphi\big(B_{r_k}(x_k)\big) \subset V$ for every $k \in \mathbb{N}$, by applying the previous inequality in (1.4) we obtain

$$\frac{\mathcal{L}^n\big(\Omega \cap B_{r_k}(x_k)\big)}{r_k^n} \geqslant \frac{\omega_n}{2M} > 0, \qquad \forall\, k \in \mathbb{N}.$$

This again contradicts (1.2) and the statement follows. □

The argument that we have used in Remark 1.1 can be adapted almost straightforwardly (by using the coarea formula for Lipschitz maps) in order to conclude that every open set $\Omega \subset\subset \mathbb{R}^n$ with Lipschitz regular boundary is still of type A, for some $A > 0$.
Nevertheless, even for relatively compact open domains in $\mathbb{R}^n$, severe issues can happen when we drop any boundary regularity requirement or if we work with unbounded domains. For instance, the presence of cusps at the boundary can be a source of troubles, as we can easily see thanks to the following example.

Example 1.1. Let $\Omega \subset\subset \mathbb{R}^2$ be given by

$$\Omega := \big\{x = (x_1, x_2) \in \mathbb{R}^2 \text{ s.t. } x_2 > \sqrt{|x_1|} \text{ and } x_2 \in (0,1)\big\}.$$

We can check by hand that this domain is not of type A, for any fixed $A > 0$. Indeed, it sufficies to find a sequence of points $\{x_k\}_{k\in\mathbb{N}} \subset \Omega$ and a sequence of radii $\{r_k\}_{k\in\mathbb{N}} \in \big(0, r_0(\Omega)\big)$ such that

$$\frac{\mathcal{L}^2\big(\Omega \cap B_{r_k}(x_k)\big)}{r_k^2} \to 0^+, \tag{1.5}$$

as $k \to +\infty$.
First, notice that

$$\begin{aligned} \frac{\mathcal{L}^2\big(\Omega \cap B_\rho(0)\big)}{\rho^2} &= \frac{2}{\rho^2}\int_0^{x_\rho}\Big(\sqrt{\rho^2 - x_1^2} - \sqrt{x_1}\Big)\, dx_1 \\ &= \frac{1}{\rho^2}\left[x_1\sqrt{\rho^2 - x_1^2} + \rho^2 \arctan\left(\frac{x_1}{\sqrt{\rho^2 - x_1^2}}\right)\right]_0^{x_\rho} - \frac{4}{3\rho^2}\left[x_1^{3/2}\right]_0^{x_\rho} \\ &= \arctan(\sqrt{x_\rho}) - \frac{x_\rho}{\rho^2}\frac{\sqrt{x_\rho}}{3} \end{aligned}$$

where

$$x_\rho := \frac{\sqrt{1+4\rho^2}-1}{2},$$

for every $0 < \rho < 1$. Since

$$\lim_{\rho \to 0^+} x_\rho = 0 \quad \text{and} \quad \lim_{\rho \to 0^+} \frac{x_\rho}{\rho^2} = 1,$$

it follows that

$$\lim_{\rho \to 0^+} \frac{\mathcal{L}^2\big(\Omega \cap B_\rho(0)\big)}{\rho^2} = 0. \tag{1.6}$$

Choose $x_k := (0, 2^{-k})$ and $r_k := k^{-1}$, for every $k \in \mathbb{N}$. Set $\rho_k := |x_k| + r_k = 2^{-k} + k^{-1}$, for every $k \in \mathbb{N}$ and notice that

$$\begin{aligned} \frac{\mathcal{L}^2\big(\Omega \cap B_{r_k}(x_k)\big)}{r_k^2} \leqslant \frac{\mathcal{L}^2\big(\Omega \cap B_{\rho_k}(0)\big)}{r_k^2} &= \left(\frac{\rho_k}{r_k}\right)^2 \frac{\mathcal{L}^2\big(\Omega \cap B_{\rho_k}(0)\big)}{\rho_k^2} \\ &= \left(\frac{2^{-k}}{k^{-1}} + 1\right)^n \frac{\mathcal{L}^2\big(\Omega \cap B_{\rho_k}(0)\big)}{\rho_k^2}. \end{aligned}$$

By equation (1.6) and since $\rho_k \to 0^+$ as $k \to +\infty$, (1.5) follows.

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