We recall the spectral decomposition theorem for the Laplacian:

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^2. Then there exists a $L^2(\Omega)$-orthonormal Hilbert basis $(\varphi_k)_{k\in\mathbb{N}}$ of $L^2(\Omega)$ and a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of positive reals satisfying $\lambda_k \to \infty$ as $k \to \infty$ such that for every $k \in \mathbb{N}$*

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } \Omega, \\ \quad\varphi_k = 0 & \text{on } \partial\Omega. \end{cases}$$

The number λ_k is called k-th *Dirichlet eigenvalue* of the Laplace operator $-\Delta$ and φ_k is called the corresponding eigenfunction. The following Lemma gives a characterisation of λ_k provided the first $k-1$ eigenfunctions $\varphi_1, \ldots, \varphi_{k-1}$ are all known.

Lemma 1. *Let $(\varphi_k)_{k\in\mathbb{N}}$ be the $L^2(\Omega)$-orthonormal basis of eigenfunctions of $-\Delta$ with corresponding Dirichlet eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ from above. Then,*

$$\lambda_k = \inf_{u\in Y_{k-1}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} = \sup_{u\in \operatorname{span}\{\varphi_1,\ldots,\varphi_k\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}},$$

where $Y_{k-1} = \{u \in H^1_0(\Omega) : \int_\Omega u\varphi_j \, dx = 0 \ \forall j = 1, \ldots, k-1\}$.

Proof. Let $\langle\cdot,\cdot\rangle$ denote the standard scalar product in $L^2(\Omega)$. Recall that for any $u \in L^2(\Omega)$ the Fourier series $(u_m)_{m\in\mathbb{N}}$ given by

$$u_m = \sum_{j=1}^{m} \langle u, \varphi_j\rangle\varphi_j$$

converges to u in $L^2(\Omega)$ as $m \to \infty$ and $\|u\|^2_{L^2(\Omega)}$ is given by the Parseval identity

$$\|u\|^2_{L^2(\Omega)} = \sum_{j\in\mathbb{N}} |\langle u, \varphi_j\rangle|^2.$$

If additionally $u \in H^1_0(\Omega)$, then orthogonality $(u - u_m) \perp u_m$ holds not only in $L^2(\Omega)$ but also in $H^1(\Omega)$. Indeed, since $\langle\nabla\varphi_j, \nabla\varphi_i\rangle = \langle-\Delta\varphi_j, \varphi_i\rangle = \lambda_j\langle\varphi_j, \varphi_i\rangle = 0$ for $j \neq i$,

$$\begin{aligned} \langle\nabla u - \nabla u_m, \nabla u_m\rangle &= \sum_{j=1}^{m} \langle u, \varphi_j\rangle\langle\nabla u, \nabla\varphi_j\rangle - \sum_{j=1}^{m} \langle u, \varphi_j\rangle^2\langle\nabla\varphi_j, \nabla\varphi_j\rangle \\ &= \sum_{j=1}^{m} \langle u, \varphi_j\rangle\langle u, \varphi_j\rangle\lambda_j - \sum_{j=1}^{m} \langle u, \varphi_j\rangle^2\lambda_j = 0. \end{aligned}$$

By Pythagoras' theorem, $\|\nabla u\|^2_{L^2(\Omega)} = \|\nabla u - \nabla u_m\|^2_{L^2(\Omega)} + \|\nabla u_m\|^2_{L^2(\Omega)}$. In particular, $\|\nabla u_m\|^2_{L^2(\Omega)} \leq \|\nabla u\|^2_{L^2(\Omega)}$ for every $m \in \mathbb{N}$. Therefore, the sequence $(u_m)_{m\in\mathbb{N}}$ is bounded in $H^1(\Omega)$. Hence a subsequence $(u_m)_{m\in\Lambda\subset\mathbb{N}}$ converges weakly in $H^1(\Omega)$ and its weak limit must be u since $(u_m)_{m\in\mathbb{N}}$ also converges in $L^2(\Omega)$ to u. Consequently,

$$\|\nabla u\|^2_{L^2(\Omega)} = \lim_{m\to\infty} \langle \nabla u_m, \nabla u\rangle = \lim_{m\to\infty} \sum_{j=1}^{m} \langle u, \varphi_j\rangle \langle \nabla \varphi_j, \nabla u\rangle = \sum_{j\in\mathbb{N}} \lambda_j |\langle u, \varphi_j\rangle|^2.$$

If additionally $u \in Y_{k-1}$, then

$$\|\nabla u\|^2_{L^2(\Omega)} = \sum_{j\geq k} \lambda_j |\langle u, \varphi_j\rangle|^2 \geq \lambda_k \sum_{j\geq k} |\langle u, \varphi_j\rangle|^2 = \lambda_k \|u\|^2_{L^2(\Omega)},$$

where equality occurs if and only if u is a multiple of φ_k. This proves the first identity. For the second identity, we notice that any $u \in \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}$ satisfies

$$\|\nabla u\|^2_{L^2(\Omega)} = \sum_{j\leq k} \lambda_j |\langle u, \varphi_j\rangle|^2 \leq \lambda_k \sum_{j\leq k} |\langle u, \varphi_j\rangle|^2 = \lambda_k \|u\|^2_{L^2(\Omega)},$$

where equality occurs if (and only if) u is a multiple of φ_k. This proves the claim. $\square$

The following Theorem provides a characterisation of λ_k which does not require knowledge of eigenfunctions. Instead it involves a second layer of minimisation.

Theorem 2 (Courant–Fischer min-max principle)**.** *The k-th Dirichlet eigenvalue of the Laplace operator $-\Delta$ is given by*

$$\lambda_k = \inf_{\substack{V\subset H^1_0(\Omega),\\ \dim V = k}} \sup_{u\in V\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

Proof. Since $\operatorname{span}\{\varphi_1, \ldots, \varphi_k\}$ is a k-dimensional subspace of $H^1_0(\Omega)$, the second identity in Lemma 1 implies

$$\lambda_k \geq \inf_{\substack{V\subset H^1_0(\Omega),\\ \dim V = k}} \sup_{u\in V\setminus\{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

Conversely, if $V \subset H^1_0(\Omega)$ is any k-dimensional subspace, then there exists an element $0 \neq w \in (V \cap Y_{k-1})$ because Y_{k-1} is of codimension $k-1$. Therefore, by the first identity of Lemma 1,

$$\lambda_k \leq \frac{\|\nabla w\|^2_{L^2(\Omega)}}{\|w\|^2_{L^2(\Omega)}} \leq \sup_{u\in V} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}.$$

Since the k-dimensional subspace $V \subset H_0^1(\Omega)$ is arbitrary, we obtain

$$\lambda_k \leq \inf_{\substack{V \subset H_0^1(\Omega), \\ \dim V = k}} \sup_{u \in V \setminus \{0\}} \frac{\|\nabla u\|^2_{L^2(\Omega)}}{\|u\|^2_{L^2(\Omega)}}$$

which completes the proof. $\square$