NOWHERE DIFFERENTIABLE SOBOLEV FUNCTIONS IN MULTIPLE DIMENSIONS

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Let $\Omega \subset \mathbb{R}^n$ be open. If $p > n$, every function $u \in W^{1,p}_{loc}(\Omega)$ is $\mathcal{L}^n$-a.e. classically differentiable on Ω (see Satz 8.6.9). We wonder about what happens in case $p \in [1, n]$, i.e. we ask ourselves the following question: are there functions in $W^{1,p}(\Omega)$ with $p \in [1, n]$ which are not classically differentiable on a subset $S \subset \Omega$ such that $\mathcal{L}^n(S) > 0$?
It turns out that the answer to such question is positive, for any dimension $n \geqslant 2$.

Remark 1.1. Notice that the answer to our question is trivially negative in case $n = 1$. Indeed, given any open interval $I \subset \mathbb{R}$ and any $p \in [1, +\infty]$, it is well known that every function $u \in W^{1,p}(I)$ is absolutely continuous. Since absolutely continuous functions on I are $\mathcal{L}^1$-a.e. differentiable, we conclude that every Sobolev function on I is indeed $\mathcal{L}^1$-a.e. differentiable.

In multiple dimensions, the situation can be much wilder and there is no upper bound on the measure of the singular set of functions in $W^{1,p}(\Omega)$ when $p \in [1, n]$. Indeed, we can show what follows.

Proposition 1.1. *Fix any $n \in \mathbb{N}$ such that $n \geqslant 2$ and let $B := B_1(0) \subset \mathbb{R}^n$ be the open unit ball in $\mathbb{R}^n$. Then, there exists a function $u \in W^{1,n}(B)$ such that u is not differentiable at any point in B.*

Proof. Consider the function $u_0 : \mathbb{R}^n \to \mathbb{R}$ given by

$$u_0(x) := \begin{cases} \log\big(\log(|x|^{-1})\big) & x \in B_{1/e}(0), \\ 0 & x \in \mathbb{R}^n \smallsetminus \overline{B_{1/e}(0)}. \end{cases}$$

A direct computation shows that $u_0 \in W^{1,n}(\mathbb{R}^n)$, see also Beispiel 8.1.2-(ii). Consider any enumeration $\{q_j\}_{j\in\mathbb{N}}$ of the countable set $\mathbb{Q} \cap B$ and, for every $j \in \mathbb{N}$, define $u_j : \mathbb{R}^n \to \mathbb{R}$ by $u_j(x) := j^{-2} u_0(x - q_j)$, for every $x \in \mathbb{R}^n$.
We notice that

$$\begin{aligned} \sum_{j=k}^{\infty} ||u_j||_{L^n(B)} &= \sum_{j=k}^{\infty} \frac{1}{j^2} \bigg(\int_B |u_0(x-q_j)|^n \, d\mathcal{L}^n(x) \bigg)^{1/n} \\ &\leqslant \sum_{j=k}^{\infty} \frac{1}{j^2} \bigg(\int_{\mathbb{R}^n} |u_0|^n \, d\mathcal{L}^n \bigg)^{1/n} \leqslant ||u_0||_{L^n(\mathbb{R}^n)} \sum_{j=k}^{\infty} \frac{1}{j^2} \xrightarrow{k\to\infty} 0^+. \end{aligned}$$

Hence, the series

$$\sum_{j=0}^{+\infty} u_j$$

is absolutely convergent in the Banach space $L^n(B)$, which implies that it converges in $L^n(B)$ to some limit $u \in L^n(B)$. Since we can always extract a subsequence having the following property, without losing generality we assume that

$$\sum_{j=0}^{k} u_j \xrightarrow{k\to\infty} u, \qquad \mathcal{L}^n\text{-a.e. on } B.$$

Analogously,

$$\sum_{j=k}^{\infty} ||\nabla u_j||_{L^n(B)} = \sum_{j=k}^{\infty} \frac{1}{j^2}\Big(\int_B |\nabla u_0(x-q_j)|^n \, d\mathcal{L}^n(x)\Big)^{1/n}$$
$$\leqslant \sum_{j=k}^{\infty} \frac{1}{j^2}\Big(\int_{\mathbb{R}^n} |\nabla u_0|^n \, d\mathcal{L}^n\Big)^{1/n} \leqslant ||\nabla u_0||_{L^n(\mathbb{R}^n)} \sum_{j=k}^{\infty} \frac{1}{j^2} \xrightarrow{k\to\infty} 0^+.$$

Hence, the series

$$\sum_{j=0}^{+\infty} \nabla u_j$$

is absolutely convergent in the Banach space $L^n(B;\mathbb{R}^n)$, which implies that it converges in $L^n(B;\mathbb{R}^n)$ to some limit $v = (v_1, ..., v_n) \in L^n(B;\mathbb{R}^n)$.
We claim that v is the weak gradient of u. Indeed, given any $m = 1, ..., n$, we get that

$$-\int_B u \frac{\partial \varphi}{\partial x_m} \, d\mathcal{L}^n = -\sum_{j=1}^{\infty} \int_B u_j \frac{\partial \varphi}{\partial x_m} \, d\mathcal{L}^n$$
$$= \sum_{j=1}^{\infty} \int_B \frac{\partial u_j}{\partial x_m} \varphi \, d\mathcal{L}^n = \int_B v_m \varphi \, d\mathcal{L}^n, \qquad \forall \varphi \in C_c^{\infty}(B).$$

Thus, $\nabla u = v \in L^n(B;\mathbb{R}^n)$ and we conclude that $u \in W^{1,n}(B)$.
Nevertheless, given any point $x \in \Omega$, u is unbounded on every ball centered at x. Indeed, if there exist a point $x \in B$ and a radius $0 < r < \operatorname{dist}(x, \partial B)$ such that u is bounded on $B_r(x)$, then

$$\int_{B_r(x)} |u_j|^{n+1} \, \mathcal{L}^n \leqslant \sum_{j=0}^{+\infty} \int_{B_r(x)} |u_j|^{n+1} = \int_{B_r(x)} |u|^{n+1} \, d\mathcal{L}^n < +\infty, \qquad \forall j \in \mathbb{N},$$

where the last equality follows by dominated convergence. But since the function $|u_j|^{n+1}$ is not integrable on balls containing q_j, this would imply that $B_r(x)$ doesn't contain any q_j, contradicting the density of $\mathbb{Q} \cap B$ in B.
Hence, u is discontinuous (and then not differentiable) at every point $x \in \Omega$. The statement follows. □

Remark 1.2. Since $B_1(0) \subset \mathbb{R}^n$ has finite $\mathcal{L}^n$-measure, clearly the function u that we have constructed in the proof of the previous proposition belongs to $W^{1,p}(B)$, for every $p \in [1, n]$. Thus, our counterexample shows that any Sobolev class $W^{1,p}(B)$ with $p \in [1, n]$ contains a nowhere differentiable function.

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