2 Strong Maximum Principle

Let $\Omega \subset\subset \mathbb{R}^n$ be an open domain and let the differential operator L be given by

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x^j} + c(x)u$$

satisfying the assumptions

(1) bounded coefficients: $a_{ij}, b_j, c \in C^0(\overline{\Omega})$,

(2) uniform ellipticity: there exists $\mu > 0$ such that

$$\forall x \in \Omega \quad \forall \xi \in \mathbb{R}^n : \quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2.$$

The goal is to complement the weak maximum principle Theorem with a local *rigidity* statement.

Theorem 1 (Strong Maximum Principle, Eberhard Hopf)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ be connected. Let $u \in C^2(\Omega)$ satisfy $Lu \geq 0$ in Ω. If $c \leq 0$ and if assumptions (1) and (2) hold, then*

$$\Big(\exists x_0 \in \Omega : \ \sup_{\Omega} u = u(x_0) \geq 0\Big) \ \Rightarrow \ u \equiv u(x_0).$$

If $c \equiv 0$, then

$$\Big(\exists x_0 \in \Omega : \ \sup_{\Omega} u = u(x_0)\Big) \qquad \Rightarrow \ u \equiv u(x_0).$$

Dropping the assumption on the sign of c,

$$\Big(\exists x_0 \in \Omega : \ \sup_{\Omega} u = u(x_0) = 0\Big) \ \Rightarrow \ u \equiv 0.$$

The key step in the proof is the following lemma.

Lemma 1 (boundary point lemma, Eberhard Hopf)**.** *Let $B := B_\rho(y) \subset \mathbb{R}^n$ and let $u \in C^2(B) \cap C^0(\overline{B})$ satisfy $Lu \geq 0$ in B with $c \leq 0$. Assume for some $x_0 \in \partial B$ that $u(x_0) \geq 0$ and $u(x) < u(x_0)$ for every $x \in B$. Then,*

$$\limsup_{h \to 0} \frac{u(x_0 + h\eta) - u(x_0)}{h} < 0,$$

where η denotes the inward-pointing unit normal of B at x_0.

If $c \equiv 0$, then the hypothesis $u(x_0) \geq 0$ can be dropped. If $u(x_0) = 0$, we can drop the assumption on the sign of the function c.

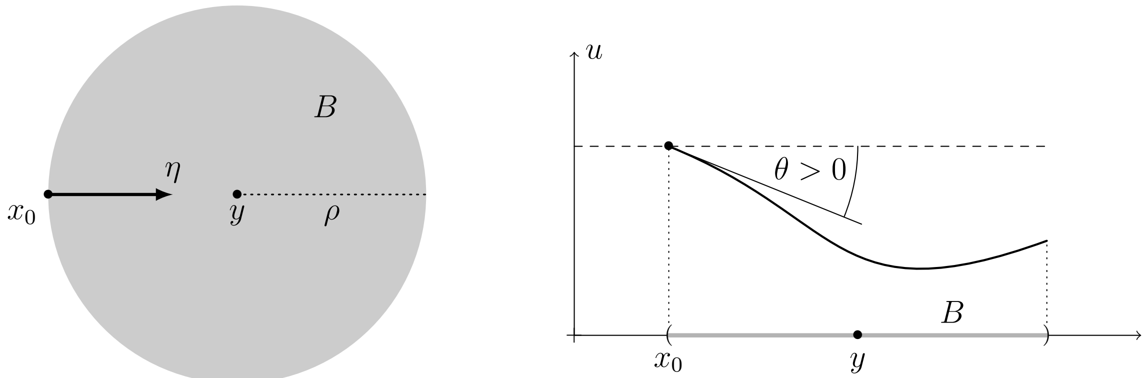

Figure 1: boundary point lemma: top view (left) and lateral view (right)

Comment. The assumption $u(x_0) > u(x)$ for all $x \in B$ implies (by basic calculus) that

$$D^+_\eta u := \limsup_{h \to 0} \frac{u(x_0 + h\eta) - u(x_0)}{h} \leq 0.$$

The point of Lemma 1 is to upgrade the weak inequality $D^+_\eta u \leq 0$ to the strict inequality $D^+_\eta u < 0$. For that, we must use the equation, i. e. the assumption $Lu \geq 0$.

Proof of Theorem 1 given Lemma 1. Let $M := \sup_\Omega u$, assume this value is attained at some $x_M \in \Omega$ and let

$$S := \{x \in \Omega : \, u(x) = M\}.$$

Since $u \in C^0(\Omega)$ its level set S is (relatively) closed in Ω. We claim that S is also (relatively) open in Ω. By contradiction, we assume the claim to be false. Hence, there exists $x_0 \in S$ and a sequence $(y_i)_{i \in \mathbb{N}}$ in $\Omega \setminus S$ such that $y_i \to x_0$ as $i \to \infty$. In particular, as shown in figure 2,

$$\exists \, \overline{y} \in \Omega \setminus S : \quad \operatorname{dist}(\overline{y}, x_0) < \operatorname{dist}(\overline{y}, \partial\Omega).$$

Moreover, by Weierstrass theorem, there exists $\overline{x} \in S$ minimizing $S \ni x \mapsto \operatorname{dist}(\overline{y}, x)$. Consequently, u satisfies the hypothesis of Lemma 1 in the ball $B_r(\overline{y})$, where $r = |\overline{x} - \overline{y}| = \operatorname{dist}(\overline{y}, S)$. Hence, $Du(\overline{x}) \neq 0$. But by the first derivative test, $Du(\overline{x}) = 0$ which is a contradiction.

Thus $S \subset \Omega$ is relatively open and closed, and certainly not empty since it contains x_M. Therefore, Ω being connected, we must conclude that $S = \Omega$ and thus u is constant. $\square$

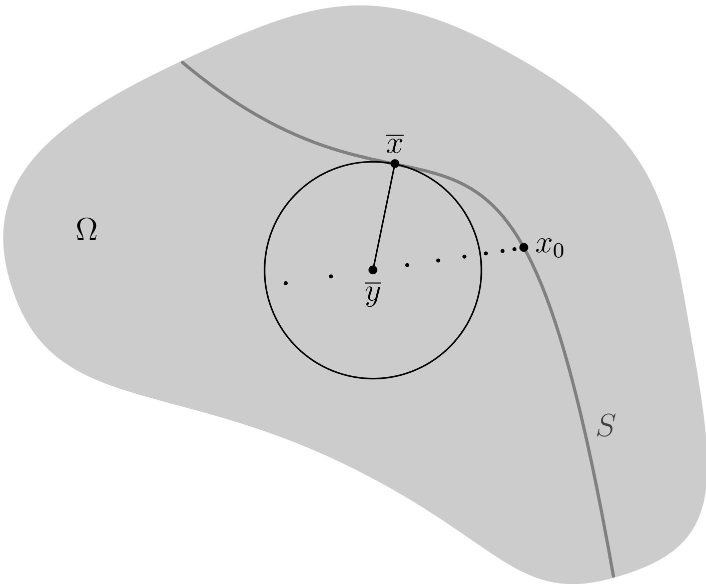

Figure 2: The setting in the proof of Theorem 1.

Proof of Lemma 1. Suppose we have already shown the general statement of the Lemma, i. e. $D_\eta^+ u < 0$ under the assumptions $c \leq 0$ and $u(x_0) \geq 0$. Let us discuss the two special clauses.

If $c \equiv 0$ and $Lu \geq 0$, then $L(u + \kappa) \geq 0$ for any $\kappa \in \mathbb{R}$. Pick $\kappa \gg 1$ such that $u + \kappa > 0$. In particular, $u(x_0) + \kappa > 0$ and the general statement yields

$$\limsup_{h \to 0} \frac{(u + \kappa)(x_0 + h\eta) - (u + \kappa)(x_0)}{h} = \limsup_{h \to 0} \frac{u(x_0 + h\eta) - u(x_0)}{h} < 0.$$

If $u(x_0) = 0$ and $u(x_0) > u(x)$ for every $x \in B$, then $u(x) \leq 0$ for every $x \in B$. Thus,

$$\begin{aligned} 0 \leq Lu &= \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{j=1}^{n} b_j(x) \frac{\partial u}{\partial x^j} + (c_+ u) - (c_- u) \\ &\leq \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x^i \partial x^j} + \sum_{j=1}^{n} b_j(x) \frac{\partial u}{\partial x^j} - (c_- u) =: L_1 u \end{aligned}$$

where $-c_-$ has the right sign, i. e. the operator L_1 satisfies the general assumptions of the Lemma, $L_1 u \leq 0$ and we obtain $D_\eta^+ u < 0$.

We proceed with the proof of the general statement of the lemma. Given $\alpha > 0$ to be determined and $r(x) := |x - y|$, we consider the function $w \colon \overline{B_\rho(y)} \to \mathbb{R}$ given by

$$w(x) = e^{-\alpha r^2} - e^{-\alpha \rho^2}.$$

We compute

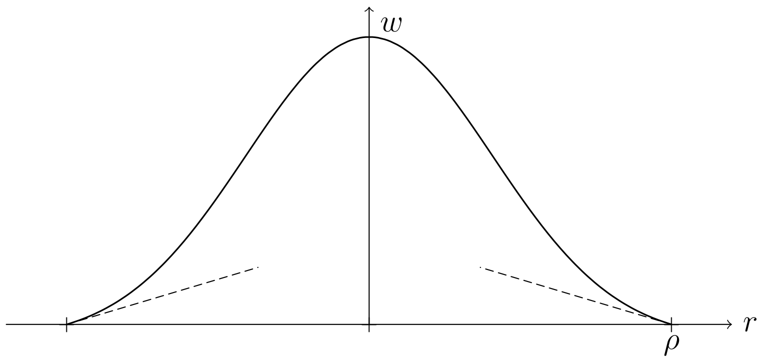

Figure 3: The graph of the function $w(r) = e^{-\alpha r^2} - e^{-\alpha\rho^2}$ for $\alpha = 3$ and $\rho = 1$.

$$Lw = e^{-\alpha r^2}\Big(4\alpha^2 \sum a_{ij}(x^i - y^i)(x^j - y^j) - 2\alpha\Big(\sum a_{ii} + \sum b_i(x^i - y^i)\Big)\Big) + cw$$

Thanks to the ellipticity assumption (with constant μ) and since $c \leq 0$, we obtain

$$Lw \geq e^{-\alpha r^2}\Big(4\alpha^2\mu r^2 - 2\alpha\Big(\sum a_{ii} + \sum b_i r\Big) + c\Big)$$

which is a quadratic polynomial in α. Therefore, $Lw > 0$ in B for $\alpha > \alpha_*$. For $0 < \varepsilon \ll 1$ we set

$$v := u - u(x_0) + \varepsilon w$$

and $A := \overline{B_\rho(y)} \setminus B_{\frac{\rho}{2}}(y)$.

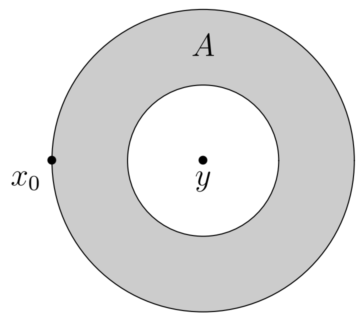

Figure 4: The set $A = \overline{B_\rho(y)} \setminus B_{\frac{\rho}{2}}(y)$

For $\varepsilon > 0$ small we have $v \leq 0$ on ∂A. Moreover,

$$Lv = \underbrace{Lu}_{\geq 0} \underbrace{- cu(x_0)}_{\geq 0} + \underbrace{\varepsilon Lw}_{>0} > 0.$$

Hence, the weak maximum principle implies $v \leq 0$ in A. By calculus,

$$\left.\begin{aligned} v(x_0) &= 0, \\ v \leq 0 &\text{ in } A \end{aligned}\right\} \qquad \Rightarrow D^+_\eta v \leq 0.$$

Consider

$$D^+_\eta v = D^+_\eta u + \varepsilon D^+_\eta w.$$

Since $D^+_\eta v \leq 0$ and $\varepsilon D^+_\eta w > 0$ we must have $D^+_\eta u < 0$. $\square$

Remark. In the setting of this lemma, if one assumes $u \in C^1(\overline{B})$ then, in particular, there exists the directional derivative

$$\frac{\partial u}{\partial \eta}$$

and its value equals (be definition) $D^+_\eta u$.

The idea behind the proof of Lemma 1 is the elliptic barrier principle. Consider $\Omega \subset\subset \mathbb{R}^n$ and $u, g \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying

$$\begin{aligned} Lu &\geq 0 &&\text{in } \Omega \\ Lg &\leq 0 &&\text{in } \Omega \\ g &\geq u &&\text{on } \partial\Omega \end{aligned}$$

If $c \leq 0$, then the weak maximum principle applied to $u - g$ yields $g \geq u$ in Ω.

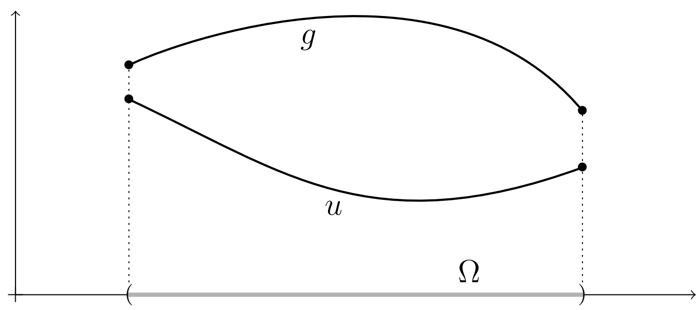

Figure 5: elliptic barrier principle

Back to Hopf: Let us revisit the argument above, and rephrase it with the language of barriers.

The goal is $v \leq 0$ on A which is equivalent to $\varepsilon w \leq u(x_0) - u(x)$ on A.

Barrier principle: Choosing $0 < \varepsilon \ll 1$ we can achieve

$$\varepsilon w \leq u(x_0) - u \quad \text{ on } \partial A$$

i. e. we can push the graph of εw below $u(x_0) - u$ such that it serves as a barrier.

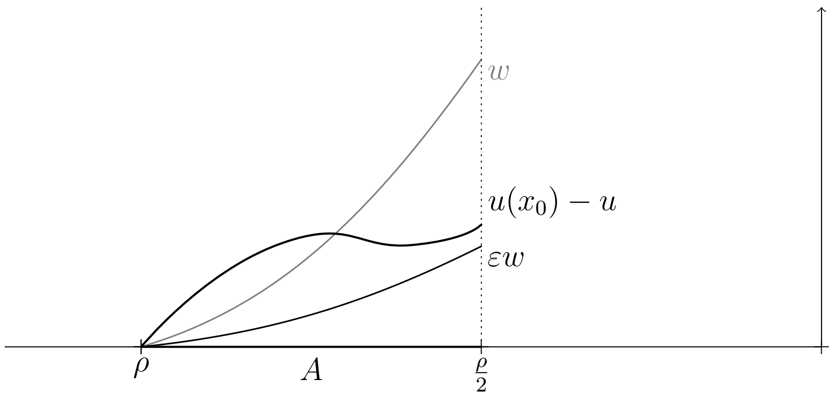

Figure 6: If $\varepsilon > 0$ is sufficiently small, then $\varepsilon w \leq u(x_0) - u$ on ∂A.

2.1 Method of sub- and supersolutions

Let $\Omega \subset\subset \mathbb{R}^n$. Consider the non-linear problem

$$\begin{cases} Lu = G(x,u) & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases} \tag{$*$}$$

We assume $\psi \in C^{2,\alpha}(\overline{\Omega})$ and that L is elliptic with coefficients $a_{ij}, b_j, c \in C^{0,\alpha}(\overline{\Omega})$. Finally, we further require $G\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ to be C^1. If you can find

- A subsolution φ^-, i. e. $\varphi^- \in C^{2,\alpha}(\overline{\Omega})$ satisfying

$$\begin{cases} L\varphi^- \geq G(x,\varphi^-) & \text{in } \Omega, \\ \varphi^- \leq \psi & \text{on } \partial\Omega, \end{cases}$$

- and a supersolution φ^+, i. e. $\varphi^+ \in C^{2,\alpha}(\overline{\Omega})$ satisfying

$$\begin{cases} L\varphi+ \leq G(x,\varphi^+) & \text{in } \Omega, \\ \varphi^+ \geq \psi & \text{on } \partial\Omega. \end{cases}$$

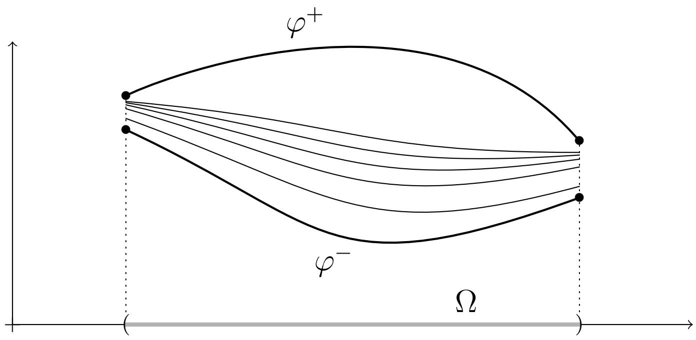

Figure 7: Approximating a solution by a sequence of subsolutions.

Then there exists a classical solution $u \in C^{2,\alpha}(\overline{\Omega})$ to problem $(*)$.

The proof of this fact is actually a relatively simple application of the barrier principle: starting with $u_0 = \varphi^-$ one constructs, by solving linear problems, a monotone sequence of functions that are uniformly bounded by φ^+, and thus must convergence to a fixed point of the iteration, which will be a solution of $(*)$. We leave the details as an (optional, but highly instructive) exercise.

2.2 Application: The Kazdan–Warner problem

Let (Σ, g) be a surface of genus γ with Gauss curvature K_g given by

$$K_g = \begin{cases} +1, & \text{if } \gamma = 0, \\ 0, & \text{if } \gamma = 1, \\ -1, & \text{if } \gamma \geq 2, \end{cases}$$

and consider the conformal metric $\tilde{g} = e^{2u} g$ on Σ which has Gauss curvature

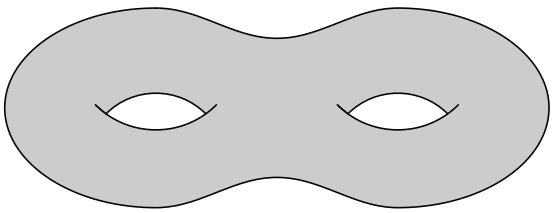

Figure 8: A surface (Σ, g) of genus $\gamma = 2$

$$K_{\tilde{g}} := e^{-2u}(K_g - \Delta_g u).$$

Question: Can we realise $K_{\tilde{g}} = f \in C^\infty(\Sigma)$? This corresponds to solving

$$G(x,u) := e^{2u} f - c = -\Delta_g u$$

Many of the things we know about this very important problem are actually obtained with the method of sub- and super-solutions.

Remark. This problem is still open in full generality if $\Sigma \simeq \mathbb{S}^2$.