WEYL LAW FOR THE LAPLACIAN

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Let $\Omega \subset\subset \mathbb{R}^n$ be open, of class C^2. We already know that the Dirichlet eigenvalues of the Laplacian on Ω form a positive and non-decreasing sequence $0 < \lambda_1 \leqslant \lambda_2 \leqslant ...$ such that $\lambda_j \to +\infty$ as $k \to +\infty$. Moreover, the following variational characterization of the j-th eigenvalue holds:

$$\lambda_j = \inf_{\substack{V \subset H_0^1(\Omega) \\ \dim(V)=j}} \sup_{V \smallsetminus \{0\}} \frac{||\nabla u||^2_{L^2(\Omega)}}{||u||^2_{L^2(\Omega)}}, \qquad \forall\, j \in \mathbb{N}_* = \{1, 2, 3 \ldots\}. \tag{1.1}$$

Remark 1.1. Recall that Dirichlet eigenfunctions of the Laplacian associated to different eigenvalues are L^2-orthogonal. This follows from the statement of the 'abstract' spectral theorem for compact self-adjoint operators, but let us see the point in very concrete terms. Indeed, let $\Omega \subset\subset \mathbb{R}^n$ be any open subset of $\mathbb{R}^n$. Assume that $\lambda_1, \lambda_2 \in \mathbb{R}$ are two different Dirichlet eigenvalues of the Laplacian on Ω and let $\varphi_1, \varphi_2 \in H_0^1(\Omega) \smallsetminus \{0\}$ be eigenfunctions of the Laplacian associated to the eigenvalues λ_1 and λ_2 respectively, i.e.

$$\int_\Omega \nabla\varphi_1 \cdot \nabla v \, d\mathcal{L}^n = \lambda_1 \int_\Omega \varphi_1 v \, d\mathcal{L}^n = \lambda_1 (\varphi_1, v)_{L^2(\Omega)},$$
$$\int_\Omega \nabla\varphi_2 \cdot \nabla v \, d\mathcal{L}^n = \lambda_2 \int_\Omega \varphi_2 v \, d\mathcal{L}^n = \lambda_2 (\varphi_2, v)_{L^2(\Omega)},$$

for every $v \in H_0^1(\Omega)$. Thus, choosing $v = \varphi_2$ (respectively: $v = \varphi_1$) in the first (respectively: second) equality one gets

$$(\lambda_1 - \lambda_2)(\varphi_1, \varphi_2)_{L^2(\Omega)} = \int_\Omega \nabla\varphi_1 \cdot \nabla\varphi_2 \, d\mathcal{L}^n - \int_\Omega \nabla\varphi_2 \cdot \nabla\varphi_1 \, d\mathcal{L}^n = 0,$$

form which it follows that $(\varphi_1, \varphi_2)_{L^2(\Omega)} = 0$.

We aim to study the asymptotic growth rate of the sequence $\{\lambda_j\}_{j \in \mathbb{N}_*}$ as $j \to +\infty$, i.e. we want to characterize the rate of divergence of the sequence in question. Results on this matter (of which there are many, even for typically non-linear phenomena such as the width spectrum determined by minimal cycles in a compact Riemannian manifold) are often referred to as **Weyl laws**.

Theorem 1.1 (Weyl law for general domains)**.** *Let $\Omega \subset\subset \mathbb{R}^n$ be open, of class C^2 and denote by $\{\lambda_j\}_{j \in \mathbb{N}_*}$ the Dirichlet eigenvalues of the laplacian on Ω. Then, there exists a constant $C := C(n, \Omega) > 1$ such that*

$$C^{-1} j^{2/n} \leqslant \lambda_j \leqslant C j^{2/n}, \qquad \forall\, j \in \mathbb{N}_*. \tag{1.2}$$

First, we notice that a Weyl law for the Laplacian can very easily be obtained in case $\Omega \subset \mathbb{R}^n$ is an n-dimensional open cube, in the following way.

Lemma 1.1 (Weyl law for cubes)**.** *Let $Q \subset \mathbb{R}^n$ be any open cube of centre $c = (c_1, ..., c_n) \in \mathbb{R}^n$ and side-length $2L > 0$. Let $\{\lambda_j\}_{j \in \mathbb{N}_*}$ be the Dirichlet eigenvalues of the laplacian on Q. Then, there exists a constant $C = C(n, L) > 1$ for which* (1.2) *holds true.*

Proof. We compute explicitly the Dirichlet spectrum of the Laplacian on Q. Notice that for every $k_1, ..., k_n \in \mathbb{N}_*$, the function

$$u_{k_1 \ldots k_n}(x_1, ..., x_n) = \sin\left(\frac{\pi}{L} k_1 (x_1 - c_1)\right) \cdot \ldots \cdot \sin\left(\frac{\pi}{L} k_n (x_n - c_n)\right)$$

is a Dirichlet eigenfunction of the Laplacian on Q associated to the eigenvalue $\frac{\pi^2}{L^2}(k_1^2+...+k_n^2)$. The set

$$S := \{u_{k_1...k_n} \ : \ k_1,...,k_n \in \mathbb{N}_*\}$$

is an Hilbertian basis of $L^2(Q)$ (the fact that it is an orthonormal family is a straightforward computation, while completeness is more delicate and is discussed in the Appendix A, Lemma A.1). By Remark 1.1, the sequence of eigenvalues $0<\lambda_1 \leqslant \lambda_2 \leqslant ...$ is given by ordering the set

$$\left\{\frac{\pi^2}{L^2}(k_1^2+...+k_n^2) \ : \ k_1,...,k_n \in \mathbb{N}_*\right\}.$$

In particular, note that:

(1) the multiplicity of the eigenvalue $\lambda_j = \frac{\pi^2}{L^2}(k_1^2+...+k_n^2)$ in the sequence $\{\lambda_j\}_{j\in\mathbb{N}_*}$ is equal to

$$\#\{(\ell_1,...,\ell_n)\in\mathbb{N}_*^n \ : \ \ell_1^2+...+\ell_n^2 = k_1^2+...+k_n^2\};$$

(2) the value j (indexing the eigenvalue in question) satisfies

$$N_1(k_1^2+...+k_n^2) < j \leqslant N_2(k_1^2+...+k_n^2) \tag{1.3}$$

where

$$N_1(t) := \#\{(\ell_1,...,\ell_n)\in\mathbb{N}_*^n \ : \ \ell_1^2+...+\ell_n^2 < t\},$$
$$N_2(t) := \#\{(\ell_1,...,\ell_n)\in\mathbb{N}_*^n \ : \ \ell_1^2+...+\ell_n^2 \leqslant t\},$$

for every $t\in(0,+\infty)$.

Main Claim: there exists $\alpha=\alpha(n)>0$ such that

$$\alpha^{-1}t^{n/2} \leqslant N_1(t) \leqslant N_2(t) \leqslant \alpha t^{n/2}. \tag{1.4}$$

Indeed, notice that $N_1(t)=\#(\mathbb{N}_*^n\cap S_t)$ and $N_2(t)=\#(\mathbb{N}_*^n\cap \overline{S}_t)$, where

$$S_t := (0,+\infty)^n \cap B(0,\sqrt{t}) \subset \mathbb{R}^n, \tag{1.5}$$

for any fixed $t\in(0,+\infty)$. Then, let

$$\|x\|_\infty = \max_{i=1,\dots,n}|x_i| \ (x\in\mathbb{R}^n)$$

and recall that

$$n^{-1/2}\|x\| \leqslant \|x\|_\infty \leqslant \|x\| \ \forall x\in\mathbb{R}^n$$

where $\|\cdot\|$ denotes the standard Euclidean norm.

Now, consider Q_t^1 and Q_t^2 to be the open n-dimensional cubes contained in $[0,+\infty)^n\subset\mathbb{R}^n$ with a vertex at the origin of $\mathbb{R}^n$ with side of length $\sqrt{t/n}$ and $\sqrt{t}$ respectively. The inequalities above, comparing the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ imply the inclusions

$$Q_t^1 \subset S_t \subset \overline{S}_t \subset \overline{Q_t^2}.$$

It follows that

$$\left[\sqrt{\frac{t}{n}}-1\right]^n = \#(\mathbb{N}_*^n\cap Q_1^t) \leqslant N_1(t) \leqslant N_2(t) \leqslant \#(\mathbb{N}_*^n\cap Q_2^t) = \left[\sqrt{t}\right]^n,$$

where $[\cdot]$ denotes the (lower) integer part of a non-negative real number. Since clearly

$$\frac{1}{2}s \leqslant [s-1], \text{ and } [s]\leqslant 2s \qquad \forall\, s\in(4,+\infty),$$

(although one can sharpen the first inequality quite a lot) we conclude that

$$\frac{1}{2^n n^{n/2}} t^{n/2} \leqslant N_1(t) \leqslant N_2(t) \leqslant t^{n/2} \leqslant 2^n n^{n/2} t^{n/2}.$$

By picking

$$\alpha = \alpha(n) := 2^n n^{n/2}$$

our claim follows.

By (1.3), we then obtain the estimate

$$\alpha^{-1} \left(\frac{L^2}{\pi^2} \lambda_j \right)^{n/2} \leqslant j \leqslant \alpha \left(\frac{L^2}{\pi^2} \lambda_j \right)^{n/2},$$

which inequality we can rephrase as a two-sided bound on λ_j:

$$\alpha^{-2/n} \left(\frac{\pi}{L} \right)^2 j^{2/n} \leqslant \lambda_j \leqslant \alpha^{2/n} \left(\frac{\pi}{L} \right)^2 j^{2/n}.$$

By setting

$$C(n, L) := \alpha^{2/n} \max \left\{ \frac{\pi}{L}, \frac{L}{\pi} \right\}^2$$

the statement follows. □

We now use the variational characterization (1.1) of the Dirichlet eigenvalues of the Laplacian in order to obtain the following comparison result.

Lemma 1.2. *Let $\Omega \subset\subset \mathbb{R}^n$ be open, of class C^2. Let $Q_1, Q_2 \subset \mathbb{R}^n$ be n-dimensional cubes such that $Q_1 \subset \Omega \subset Q_2$. Let $\{\lambda_j\}_{j \in \mathbb{N}_*}$, $\{\lambda_j^1\}_{j \in \mathbb{N}_*}$ and $\{\lambda_j^2\}_{j \in \mathbb{N}_*}$ be the Dirichlet eigenvalues of the Laplacian respectively on Ω, Q_1 and Q_2.*
Then,

$$\lambda_j^2 \leqslant \lambda_j \leqslant \lambda_j^1, \qquad \forall\, j \in \mathbb{N}_*.$$

Proof. Notice that $H_0^1(Q_1) \subset H_0^1(\Omega)$, in the sense every function $u \in H_0^1(Q_1)$ can be extended to a function $\tilde{u}$ in $H_0^1(\Omega)$ by simply setting $\tilde{u} \equiv 0$ on $\Omega \setminus Q_1$ (recall the homework Problem 6.2).
Analogously, $H_0^1(\Omega) \subset H_0^1(Q_2)$ and thus in fact

$$H_0^1(Q_1) \subset H_0^1(\Omega) \subset H_0^1(Q_2) \tag{1.6}$$

Fix any $j \in \mathbb{N}_*$ and denote by $\mathbb{G}_j$, $\mathbb{G}_j^1$ and $\mathbb{G}_j^2$ the set of all the j-dimensional subspaces of $H_0^1(\Omega)$, $H_0^1(Q_1)$ and $H_0^1(Q_2)$ respectively. By (1.6), we get

$$\mathbb{G}_j^1 \subset \mathbb{G}_j \subset \mathbb{G}_j^2. \tag{1.7}$$

By (1.7) and (1.1), the statement follows. □

The Weyl law for the Laplacian on general (regular enough) domains is a straightforward consequence of Lemma 1.1 and Lemma 1.2.

Appendix A. A useful Hilbertian basis of $L^2\big((-\pi,\pi)^n\big)$

Lemma A.1. *Let $Q := (-\pi,\pi)^n \subset \mathbb{R}^n$ and consider the orthonormal subset $S \subset L^2(Q)$ given by*

$$S := \big\{u(x_1,...,x_n) := \sin(k_1x_1)\cdot ... \cdot \sin(k_nx_n)\ :\ k_1,...,k_n \in \mathbb{N}_*\big\}.$$

Then,

$$L^2(Q) = \overline{\operatorname{span}(S)}.$$

Proof. When $n = 1$, the result is well-known (although highly non-trivial) and a proof can be found in [ADPM11][Proposition 5.6]. Hence, we will focus on the case $n \geqslant 2$.
Consider the set

$$\Gamma := \big\{(f_1 \otimes ... \otimes f_n)(x_1,...,x_n) := f_1(x_1)\cdot ... \cdot f_n(x_n)\ :\ f_1,...,f_n \in C^0([-\pi,\pi])\big\}.$$

We know that $\operatorname{span}(\Gamma)$ is dense in $C^0\big(\overline{Q}\big)$, by the Stone-Weierstrass theorem. Since $C^0\big(\overline{Q}\big)$ is dense in $L^2(Q)$, it is enough to prove that $\operatorname{span}(S)$ is dense in Γ.

Fix any $f_1 \otimes ... \otimes f_n \in \Gamma$. By Proposition 5.6 in [ADPM11], for every fixed $j = 1,...,n$ it holds that

$$f_j(x_j) = \sum_{h_j \in \mathbb{N}} \lambda_{h_j} \sin(k_{h_j}x_j), \qquad \text{in } L^2(-\pi,\pi),$$

for some $\{\lambda_{h_j}\}_{h_j\in\mathbb{N}} \subset \mathbb{R}$ and some $\{k_{h_j}\}_{h_j\in\mathbb{N}} \subset \mathbb{N}_*$. Moreover,

$$||f_j||^2_{L^2} = \sum_{h_j\in\mathbb{N}} \lambda^2_{h_j},$$

since the set $\{\sin(k_{h_j}x_j)\}_{h_j\in\mathbb{N}}$ is orthonormal in $L^2(-\pi,\pi)$. We claim that

$$(f_1 \otimes ... \otimes f_n)(x_1,...,x_n) = \sum_{h_1,...,h_n\in\mathbb{N}} \lambda_{h_1}...\lambda_{h_n} \sin(k_{h_1}x_1)\cdot ... \cdot \sin(k_{h_n}x_n),$$

in $L^2(Q)$. The previous claim will imply that $\operatorname{span}(S)$ is dense in Γ and the statement will follow.

In order to prove the claim, we proceed by induction on $n \geqslant 2$.

Basis of the induction: we assume $n = 2$ and we want to show that

$$f_1 \otimes f_2 = \sum_{h_1,h_2\in\mathbb{N}_*} \lambda_{h_1}\lambda_{h_2} \sin(k_{h_1}x_1)\sin(k_{h_2}x_2), \tag{A.1}$$

in $L^2(Q)$. For every $m \in \mathbb{N}$, we define

$$s_m(x_1) := \sum_{h_1=1}^{m} \lambda_{h_1} \sin(k_{h_1}x_1), \qquad \forall\, x_1 \in (-\pi,\pi),$$
$$q_m(x_2) := \sum_{h_2=1}^{m} \lambda_{h_2} \sin(k_{h_2}x_2), \qquad \forall\, x_2 \in (-\pi,\pi).$$

We notice that

$$(s_m \otimes q_m)(x_1,x_2) = \sum_{h_1,h_2=1}^{m} \lambda_{h_1}\lambda_{h_2} \sin(k_{h_1}x_1)\sin(k_{h_2}x_2), \qquad \forall\,(x_1,x_2) \in Q.$$

Moreover,

$$||s_m||^2_{L^2} = \int_{-\pi}^{\pi} |s_m|^2\, d\mathcal{L}^1 = \int_{-\pi}^{\pi} \left|\sum_{h_1=0}^{m} \lambda_{h_1} \sin(k_{h_1}x_1)\right|^2 d\mathcal{L}^1(x_1)$$
$$= \sum_{h_1=0}^{m} \int_{-\pi}^{\pi} \lambda^2_{h_1} \sin(k_{h_1}x_1)^2\, d\mathcal{L}^1(x_1) = \sum_{h_1=0}^{m} \lambda^2_{h_1} \leqslant ||f_1||^2_{L^2}$$

where the last two equalities follow since the set $\{\sin(k_{h_1}x_1)\}_{h_1\in\mathbb{N}}$ is orthonormal in $L^2(-\pi,\pi)$. Then, we compute

$$\begin{aligned}\int_Q |f_1\otimes f_2 - s_m\otimes q_m|^2\,d\mathcal{L}^2 &\leqslant 2\int_Q |f_1-s_m|^2|f_2|^2\,d\mathcal{L}^2 + 2\int_Q |s_m|^2|f_2-q_m|^2\,d\mathcal{L}^2\\ &= 2||f_2||_{L^2}\int_Q |f_1-s_m|^2\,d\mathcal{L}^2 + 2||s_m||^2_{L^2}\int_Q |f_2-q_m|^2\,d\mathcal{L}^2\\ &= 2||f_2||_{L^2}\int_Q |f_1-s_m|^2\,d\mathcal{L}^2 + 2||f_1||^2_{L^2}\int_Q |f_2-q_m|^2\,d\mathcal{L}^2 \xrightarrow{m\to\infty} 0^+.\end{aligned}$$

Thus, the basis of the induction is proved.

Inductive step: by induction, we get that

$$f_1\otimes\ldots\otimes f_{n-1} \sum_{h_1,\ldots,h_{n-1}\in\mathbb{N}} \lambda_{h_1}\ldots\lambda_{h_{n-1}}\sin(k_{h_1}x_1)\cdot\ldots\cdot\sin(k_{h_{n-1}}x_{n-1}),$$

in $L^2\big((-\pi,\pi)^{n-1}\big)$. Moreover, since the set $\{\sin(k_{h_1}x_1)\cdot\ldots\cdot\sin(k_{h_{n-1}}x_{n-1})\}_{h_1,\ldots,h_{n-1}\in\mathbb{N}}$ is orthonormal in $L^2\big((-\pi,\pi)^{n-1}\big)$, we get

$$||f_1\otimes\ldots\otimes f_{n-1}||^2_{L^2} = \sum_{h_1,\ldots,h_{n-1}\in\mathbb{N}} \lambda^2_{h_1}\ldots\lambda^2_{h_{n-1}}.$$

For every $m\in\mathbb{N}$, we define

$$\begin{aligned} s_m(x_1,\ldots,x_{n-1}) &:= \sum_{h_1,\ldots,h_{n-1}=0}^{m} \lambda_{h_1}\ldots\lambda_{h_{n-1}}\sin(k_{h_1}x_1)\cdot\ldots\cdot\sin(k_{h_{n-1}}x_{n-1})\\ q_m(x_n) &:= \sum_{h_n=0}^{m}\lambda_{h_n}\sin(k_{h_n}x_n).\end{aligned}$$

for all $(x_1,\ldots,x_{n-1})\in(-\pi,\pi)^{n-1}$ and $x_n\in(-\pi,\pi)$.
We notice that

$$(s_m\otimes q_m)(x_1,\ldots,x_n) = \sum_{h_1,\ldots,h_n=0}^{m}\lambda_{h_1}\ldots\lambda_{h_n}\sin(k_{h_1}x_1)\cdot\ldots\cdot\sin(k_{h_n}x_n), \qquad \forall\,(x_1,\ldots,x_n)\in Q.$$

Moreover,

$$\begin{aligned} ||s_m||^2_{L^2} &= \int_{(-\pi,\pi)^{n-1}} |s_m|^2\,d\mathcal{L}^{n-1}\\ &= \int_{(-\pi,\pi)^{n-1}}\left|\sum_{h_1,\ldots,h_{n-1}=0}^{m}\lambda_{h_1}\ldots\lambda_{h_{n-1}}\sin(k_{h_1}x_1)\cdot\ldots\cdot\sin(k_{h_{n-1}}x_{n-1})\right|^2 d\mathcal{L}^{n-1}(x_1,\ldots,x_{n-1})\\ &= \sum_{h_1,\ldots,h_{n-1}=0}^{m}\int_{(-\pi,\pi)^{n-1}}\lambda^2_{h_1}\ldots\lambda^2_{h_{n-1}}\sin(k_{h_1}x_1)^2\cdot\ldots\cdot\sin(k_{h_{n-1}}x_{n-1})^2\,d\mathcal{L}^{n-1}(x_1,\ldots,x_{n-1})\\ &= \sum_{h_1=0}^{m}\lambda^2_{h_1}\ldots\lambda^2_{h_{n-1}} \leqslant ||f_1\otimes\ldots\otimes f_{n-1}||^2_{L^2}.\end{aligned}$$

where the last two equalities follow since the set $\{\sin(k_{h_1}x_1)\cdot\ldots\cdot\sin(k_{h_{n-1}}x_{n-1})\}_{h_1,\ldots,h_{n-1}\in\mathbb{N}}$ is orthonormal in $L^2((-\pi,\pi)^{n-1})$.

Then, we compute

$$\begin{aligned}
\int_Q |f_1 \otimes ... \otimes f_n - s_m \otimes q_m|^2 \, d\mathcal{L}^n &\leqslant 2 \int_Q |f_1 \otimes ... \otimes f_{n-1} - s_m|^2 |f_n|^2 \, d\mathcal{L}^n \\
&\quad + 2 \int_Q |s_m|^2 |f_n - q_m|^2 \, d\mathcal{L}^n \\
&= 2||f_n||_{L^2} \int_Q |f_1 \otimes ... \otimes f_{n-1} - s_m|^2 \, d\mathcal{L}^n \\
&\quad + 2||s_m||_{L^2}^2 \int_Q |f_n - q_m|^2 \, d\mathcal{L}^n \\
&\leqslant 2||f_2||_{L^2} \int_Q |f_1 - s_n|^2 \, d\mathcal{L}^n \\
&\quad + 2||f_1 \otimes ... \otimes f_{n-1}||_{L^2}^2 \int_Q |f_2 - q_n|^2 \, d\mathcal{L}^n \xrightarrow{m \to \infty} 0^+.
\end{aligned}$$

Hence, the inductive step is proved and the statement follows. □

References

[ADPM11] Luigi Ambrosio, Giuseppe Da Prato, and Andrea Mennucci. *Introduction to measure theory and integration.* Vol. 10. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, 2011, pp. xii+187.

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