Functional Analysis II — L22 — 17/05/2021

Estimates for weak sols. of inhomogeneous elliptic pdes w/ constant coefficients

$$A^{(0)} u := -\frac{\partial}{\partial x_i}\left(a_{ij}^{(0)}(x) \frac{\partial u}{\partial x_j} \right) = -\frac{\partial f_i}{\partial x_i} + h \qquad (*)$$

($a_{ij}^{(0)}$: constant coefficients)

Setup: $B_R^{(+)}(0)$ may mean

interior case $B_R(0)$ boundary case $B_R^+(0)$

today's argument is identical in both cases

- $u \in H^1(B_R^{(+)}(0))$, w/ additional hp. that $\mathrm{tr}(u) = 0$ on $\partial\mathbb{R}^n_+ \cap B_R(0)$ if in the boundary case

weak sol to $(*)$ for given data $h \in L^2(B_R^{(+)}(0))$, $f_i \in L^2(B_R^{(+)}(0))\ \forall i \in \{1,\dots,n\}$

Lemma (inhomogeneous Caccioppoli estimates)

- $$\int_{B_r^{(+)}(0)} |\nabla u|^2 \le C\left(\frac{r}{R}\right)^n \int_{B_R^{(+)}} |\nabla u|^2 + C\int_{B_R^{(+)}} |f_i - (\overline{f_i})_R|^2 + CR^2 \int_{B_R^{(+)}} |h|^2$$

- $$\int_{B_r^{(+)}(0)} |\nabla u - \overline{(\nabla u)}_r|^2 \le C\left(\frac{r}{R}\right)^{n+2} \int_{B_R^{(+)}(0)} |\nabla u - \overline{(\nabla u)}_R|^2 + C\int_{B_R^{(+)}(0)} |f_i - \overline{(f_i)}_R|^2 + CR^2 \int_{B_R^{(+)}(0)} |h|^2$$

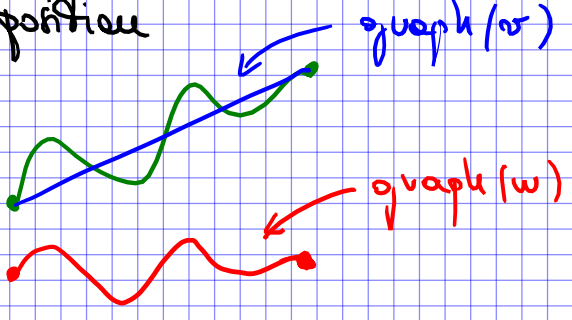
Proof it's enough to prove the 1st

(since 2nd = 1st ⊕ Poincaré-Wirtinger)

wlog. from now on I'll write $B_R(0)$ instead of $B_R^{(+)}(0)$

(key idea): orthogonal decomposition

$$\begin{cases} u = v + w \\ A^{(0)} v = 0 \\ w \in H^1_0(B_R(0)) \end{cases}$$



$$\int_{B_r(0)} |\nabla u|^2 \overset{\nabla u = \nabla v + \nabla w}{\leq} 2\int_{B_r(0)} |\nabla v|^2 + 2\int_{B_r(0)} |\nabla w|^2$$

$$\leq C\left(\frac{r}{R}\right)^4 \int_{B_R(0)} |\nabla v|^2 + C\int_{B_R(0)} |\nabla w|^2$$

$\nabla v = \nabla u - \nabla w$

$$\leq C\left(\frac{r}{R}\right)^4 \int_{B_R(0)} |\nabla u|^2 + C\int_{B_R(0)} |\nabla w|^2 \qquad (\$)$$

now need to use equations for $w \in H^1_0(B_R(0))$

$$A^{(0)} w = -\frac{\partial f_i}{\partial x_i} + h$$

$$\lambda \|\nabla w\|^2_{L^2} \leq \int_{B_R(0)} (A^{(0)} w)\cdot w \overset{\text{eq.}}{=} \underbrace{\int_{B_R(0)} hw}_{①} + \underbrace{\int_{B_R(0)} f_i \frac{\partial w}{\partial x_i}}_{②}$$

$$\leq \frac{\lambda}{4}\|\nabla w\|^2_{L^2} + \frac{4cR^2}{\lambda}\int_{B_R(0)} |h|^2 \qquad ①$$

① Peter-Paul ⊕ Poincaré

$$+ \frac{\lambda}{4}\|\nabla w\|^2_{L^2} + \frac{4}{\lambda}\int_{B_R(0)} |f_i - (\overline{f_i})_R|^2 \qquad ②$$

② Peter-Paul ⊕ $\int_{B_R(0)} \frac{\partial w}{\partial x_i}\,dx = 0$ $(\forall i)$ and so $\int_{B_R(0)} f_i \frac{\partial w}{\partial x_i} = \int_{B_R(0)} (f_i - (\overline{f_i})_R)\frac{\partial w}{\partial x_i}$

(Problem 11.2)

Absorb on the left-hand side:

$$(\in) \qquad \frac{\lambda}{2}\|\nabla w\|^2_{L^2} \leq \left(\frac{4c}{\lambda}\right) R^2 \|h\|^2_{L^2} + \frac{4}{\lambda}\int_{B_R(0)} |f_i - (\overline{f_i})_R|^2$$

So there exists $C = C(u, A^{(0)})$ such that our desired estimate follows from (\$) and (€). ∎

To get to the Schauder estimates "done right" we need to introduce 2 tools:

i) Morrey spaces

ii) Iteration lemmas

Morrey spaces

- recall: Campanato spaces

$$\mathcal{L}^{p,\lambda}(\Omega) = \left\{ f \in L^p(\Omega) : \sup_{\substack{x_0 \in \Omega \\ 0<r<r_0}} r^{-\lambda} \int_{\Omega_r(x_0)} |f - f_{x_0,r}|^p < \infty \right\}$$

$$[f]^p_{\mathcal{L}^{p,\lambda}}$$

Banach space w/ norm

$$\|f\|_{\mathcal{L}^{p,\lambda}} := \|f\|_{L^p} + [f]_{\mathcal{L}^{p,\lambda}}$$

Facts: if $\lambda > n$ then Campanato spaces are isomorphic (equiv.) to Hölder spaces $\mathcal{L}^{p,\lambda}(\Omega) \cong C^{0,\alpha}(\bar{\Omega})$

for $\alpha = \frac{\lambda - n}{p}$.

- Morrey spaces $\Omega \subset \mathbb{R}^n$ open, $\nu \geq 0$

$$L^{p,\nu}(\Omega) = \left\{ f \in L^p(\Omega) : \sup_{\substack{x_0 \in \Omega \\ 0<r<r_0}} r^{-\nu} \int_{\Omega_r(x_0)} |f|^p < \infty \right\}$$

$$[f]^p_{L^{p,\nu}}$$

Banach space, w/ norm

$$\|f\|_{L^{p,\nu}} := \|f\|_{L^p} + [f]_{L^{p,\nu}}$$

Achtung!:

$|\nu = u|$ $\quad L^{2,u} \cong L^2 \cap L^\infty$

$|\nu > u|$ $\quad L^{2,\nu} \cong \{0\}$

} Lebesgue diff. theorem

Lemma: if Ω is a type-A domain ($\Leftarrow$ class C^1) and $0 \le \nu < u$ then $L^{2,\nu}(\Omega) \cong \mathcal{L}^{2,\nu}(\Omega)$

same elements, w/ equivalent norms.

Proof: enough to prove 1) $[f]_{\mathcal{L}^{2,\nu}} \le [f]_{L^{2,\nu}}$

2) $[f]_{L^{2,\nu}} \le C \|f\|_{\mathcal{L}^{2,\nu}}$

1): follows from variance-minimizing property of the average

$$\int_{\Omega_r(x_0)} |f - f_{x_0,r}|^2 \le \int_{\Omega_r(x_0)} |f|^2 \le r^\nu [f]^2_{L^{2,\nu}}$$

multiply by $r^{-\nu}$ and get $[f]_{\mathcal{L}^{2,\nu}} \le [f]_{L^{2,\nu}}$

2): $$\int_{\Omega_\nu(x_0)} |f|^2 \le 2 \int_{\Omega_{\nu,x_0}} |f - f_{x_0,\nu}|^2 + 2 \underbrace{\int_{\Omega_{\nu,x_0}} |f_{x_0,\nu}|^2}$$

$$\le 2 \cdot r^\nu [f]^2_{\mathcal{L}^{2,\nu}} + 2 \underbrace{\int_{\Omega_{r,x_0}} |f_{x_0,r}|^2}$$

$$\cong 2 |f_{x_0,r}|^2 r^u$$

Given such $r \in (0,1)$ set $r_i = 2^i r$

and fix $i_0 \in \mathbb{N}$ such that $r_{i_0} \in [1,2)$

(wlog I'll from now on ensure $r_0 = \min\{1, \operatorname{diam}(\Omega)\} = 1$)

$$f_{x_0,r} = \underbrace{(f_{x_0,r} - f_{x_0,r_1}) + (f_{x_0,r_1} - f_{x_0,r_2}) + \dots + f_{x_0,r_{i_0}}}$$

triangle inequality

$$|f_{x_0,r}| \le \sum_{i=1}^{i_0} |f_{x_0,r_i} - f_{x_0,r_{i-1}}| + |f_{x_0,r_{i_0}}|$$

if we square

$$|f_{x_0,r}|^2 \le 2\Big(\underbrace{\sum_{i=1}^{i_0} |f_{x_0,r_i} - f_{x_0,r_{i-1}}|}_{(I)}\Big)^2 + \underbrace{2|f_{x_0,r_{i_0}}|^2}_{(II)}$$

for term (II)

$$\int_{\Omega_{r,x_0}} |f_{x_0,r_{i_0}}|^2 \le A r^u |f_{x_0,r_{i_0}}|^2 \quad \text{and}$$

$$|f_{x_0,r_{i_0}}|^2 = \Big| \fint_{\Omega_{r_{i_0}}(x_0)} f \Big|^2 \overset{\text{Jensen}}{\le} \fint_{\Omega_{r_{i_0}}(x_0)} |f|^2 \le C \|f\|_{L^2}^2$$

$$(II) \le C r^u \|f\|_{L^2}^2$$

for term (I)

$$\sum_{i=1}^{i_0} |f_{x_0,r_i} - f_{x_0,r_{i-1}}| = \sum_{i=1}^{i_0} \Big| \fint_{\Omega_{r_{i-1}}(x_0)} (f(y) - f_{x_0,r_i})\,dy \Big|$$

$$\overset{\text{Jensen}}{\le} \sum_{i=1}^{i_0} \Big(\fint_{\Omega_{r_{i-1}}(x_0)} |f(y) - f_{x_0,r_i}|^2 dy \Big)^{1/2}$$

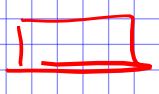
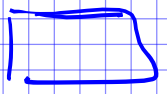
$$\overset{\text{type A}}{\le} \sum_{i=1}^{i_0} \Big(\fint_{\Omega_{r_i}(x_0)} |f(y) - f_{x_0,r_i}|^2 dy \Big)^{1/2}$$

$$\overset{r_i = r\,2^{-i}}{\le} \sum_{i=1}^{i_0} r_i^{\frac{\nu-u}{2}} [f]_{\mathcal{L}^{2,\nu}} = C [f]_{\mathcal{L}^{2,\nu}} r^{\frac{\nu-u}{2}} \sum_{i=1}^{i_0} 2^{\left(\frac{\nu-u}{2}\right) i}$$

$$\le C [f]_{\mathcal{L}^{2,\nu}} r^{\frac{\nu-u}{2}} \quad \text{for } q = 2^{\left(\frac{\nu-u}{2}\right)} \le \sum_{i=0}^{\infty} q^i$$

So $$\Big(\sum_{i=1}^{i_0} |f_{x_0,r_i} - f_{x_0,r_{i-1}}| \Big)^2 \le C r^{\nu-u} [f]_{\mathcal{L}^{2,\nu}}^2$$

term (I): $$\int_{\Omega_r(x_0)} \Big(\quad \uparrow \quad \Big) \le C r^{\nu} [f]_{\mathcal{L}^{2,\nu}}^2$$

Finally use  and  in the main estimate

$$\int_{\Omega_\nu(x_0)} |f|^2 \le 2\cdot r^\nu [f]^2_{\mathcal{L}^{2,\nu}} + C r^n \|f\|^2_{L^2} + C r^\nu [f]^2_{\mathcal{L}^{2,\nu}}$$

$$r^{-\nu} \underbrace{\int_{\Omega_\nu(x_0)} |f|^2}_{\text{uniformly bounded on } x_0 \in \Omega,\ 0<r<r_0=1} \le C\|f\|^2_{L^2} + C[f]^2_{\mathcal{L}^{2,\nu}}$$

and $[f]^2_{L^{2,\nu}} \le C\|f\|^2_{L^2} + C[f]^2_{\mathcal{L}^{2,\nu}}$ ∎

Iteration lemma: how to simplify estimates w/ many terms

Lemma Let $\phi : (0, R_0] \to [0,\infty)$ monotone non-decreasing

$$\phi(r) \le A\left(\left(\frac{r}{R}\right)^\alpha + \epsilon\right)\phi(R) + B R^\beta$$

$\forall\ 0 < r < R \le R_0$, for some $0 < \beta < \alpha$

and constants $A, B \in \mathbb{R}_{>0}$

Then $\exists\ \epsilon_0 = \epsilon_0(A,\alpha,\beta) > 0$

$\exists\ C = C(A,\alpha,\beta) > 0$ such that if $0 \le \epsilon < \epsilon_0$

$$\phi(r) \le C\left(\left(\frac{r}{R}\right)^\beta \phi(R) + B r^\beta\right)$$

In particular, can take $R = R_0 = 1$ in which case one gets

$$\phi(r) \le C\left(r^\beta \phi(1) + B r^\beta\right)$$

Pf: Exercise 11.3, Lemma 10.3.2.

Prop. (local $C^{1,\alpha}$- Schauder estimate)

Let $u \in H^1(\mathbb{R}^4)$ or $u \in H^1(\mathbb{R}^4_+)$ w/ $\mathrm{tr}|u| = 0$ on $\partial\mathbb{R}^4_+$

and (in both cases) $\mathrm{supp}(u) \subset\subset \mathbb{R}^n$, solving

$$A^{(0)} u = - \frac{\partial f_i}{\partial x_i} + h$$

for $f = (f_1, \dots, f_n) \in \mathcal{L}^{2,\mu}(\mathbb{R}^n_{(+)})$

$h \in L^{2,\nu}(\mathbb{R}^n_{(+)})$

w/ $0 < \mu = \nu + 2 < n + 2$.

Then $\theta u \in \mathcal{L}^{2,\mu}(\mathbb{R}^n_{(+)})$ w/ estimate

$$\| \theta u \|_{\mathcal{L}^{2,\mu}} \leq C \left(\| u \|_{H^1} + [f]_{\mathcal{L}^{2,\mu}} + \| h \|_{L^{2,\nu}} \right).$$

Rephrasing the conclusion:

when $\mu = \nu + 2 = n + 2\alpha$ for some $\alpha \in (0,1)$

then $\theta u \in C^{0,\alpha}$ and

$$\| \theta u \|_{C^{0,\alpha}} \leq C \left(\| u \|_{H^1} + [f]_{C^{0,\alpha}} + \underbrace{\| h \|_{L^{2,\nu}}}_{\lesssim \| h \|_{C^0}} \right)$$

Proof. (notational purposes: refer to interior case)

Set $\phi(r) := \int_{B_r(x_0)} | \theta u - \overline{(\theta u)}_r |^2$ $\left\{ \begin{array}{l} \bullet \text{ non-negative} \\ \bullet \text{ non-decreasing} \\ \quad \text{(variance-min. property ...)} \end{array} \right.$

Lemma above reads

$$\int_{B_r(x_0)} | \theta u - \overline{(\theta u)}_r |^2 \leq C \left(\frac{r}{R} \right)^{n+2} \int_{B_R(x_0)} | \theta u - \overline{(\theta u)}_R |^2 + C \int_{B_R(x_0)} | f - \overline{f}_R |^2 + C R^2 \int_{B_R(x_0)} | h |^2$$

equivalently

$$\phi(r) \leq C \left(\frac{r}{R} \right)^{n+2} \phi(R) + C R^\mu [f]^2_{\mathcal{L}^{2,\mu}} + C R^2 \cdot R^\nu [h]^2_{L^{2,\nu}}$$

$$\phi(r) \leq C \left(\frac{r}{R} \right)^{n+2} \phi(R) + \underbrace{C \left([f]^2_{\mathcal{L}^{2,\mu}} + [h]^2_{L^{2,\nu}} \right)}_{B} R^\mu$$

here $\mu < n + 2$, so apply iteration w/ $\left\{ \begin{array}{l} \alpha = n+2 \\ \beta = \mu \\ \epsilon = 0 \end{array} \right.$

$0 < r < R = 1$

$$\phi(r) \leq C r^\mu \phi(1) + C r^\mu (B)$$

∎