Functional Analysis II — 16 — 11/03/2021

ODEs on a bounded interval

$-\infty < a < b < +\infty$, $I = (a, b)$

(a) Dirichlet boundary conditions

$f \in C^0(\bar{I})$, want to solve for $u \in C^2(\bar{I})$ the problem

$$(*) \begin{cases} -u'' = f & \text{on } I \\ u(a) = u(b) = 0 & \text{on } \partial I \end{cases} \longleftarrow$$

1) weak sol. : $\exists!\ u \in H^1_0(I)$ weakly solving $(*)$ in the sense that

$$(*)_w \quad \int_I u' \cdot v' = \int_I f v \qquad \forall v \in \overline{C^\infty_c(I)}^{W^{1,2}} = H^1_0$$

2) classical sol. : we want to prove that such u belongs $C^2(\bar{I})$ and solves $(*)$ in a classical/pointwise sense.

Pf: • usually speaking we view $-u'' = f$ as a fake 1st order eq.

$$(u')' = -\underbrace{f}_{\in C^0(\bar{I}) \hookrightarrow L^\infty(I) \hookrightarrow L^2(I)}$$

i.e. $u' \in L^2$ (since $u \in H^1_0$) has a weak derivative also in $L^2(I)$ (this is the meaning of $(*)_w$ together w/ $u \in H^1_0$)

$\Longleftrightarrow$ $u' \in W^{1,2}(I) \equiv H^1(I)$

now appeal to fund. thm. of Calculus $\rightsquigarrow u' \in AC(\bar{I})$

$$u'(x) = u'(x_0) - \int_{x_0}^{x} f(t)\,dt$$

but $f \in C^0(\bar{I}) \rightsquigarrow u' \in C^1(\bar{I}) \rightsquigarrow u \in C^2(\bar{I})$

• we still need to discuss in what sense u satisfies the Dirichlet

boundary conditions, namely we want to prove $u(a) = u(b) = 0$.

How? $H^1_0(I) := \overline{C^\infty_c(I)}^{W^{1,2}}$ so $\exists\ (u_k)$ seq. in $C^\infty_c(I)$ w/ $\|u_k - u\|_{L^\infty} \leq C \|u_k - u\|_{H^1} \xrightarrow{k\to\infty} 0$

$\forall k \in \mathbb{N}$ ↑ Sobolev embedding

$|u(a)| \leq \underbrace{|u_k(a)|}_{=0} + \underbrace{\|u - u_k\|_{L^\infty}}_{\to 0,\ k\to\infty} \Longrightarrow u(a) = 0$

similarly $u(b) = 0$ ∎

(b) Neumann boundary conditions

setup as above, $f \in C^0(\bar{I})$ want to solve for $u \in C^2(\bar{I})$

$$(\#) \quad \begin{cases} -u'' + u = f \\ u'(a) = u'(b) = 0 \end{cases} \longleftarrow \text{Neumann}$$

Posing the weak problem?

⟶ $u'(a)$, $u'(b)$ are a priori not even defined if $u \in H^1$

⟶ second issue: H^1_0 not quite the right function space

Given any $v \in C^1(\bar{I})$ (no boundary conditions imposed)

take $(\#)$, multiply by v, integrate by parts (ensuring $u \in C^2(\bar{I})$ is a classical sol. to $(\#)$)

$$(\#)_w \quad \int_I u' v' + u \cdot v = \int_I f v \qquad (\forall\ v \in H^1)$$

set $(u, v)_{H^1} = \int_I (u' \cdot v' + uv)$ { standard $W^{1,2}$ norm }

$$\ell : C^1(\bar{I}) \longrightarrow \mathbb{R}$$
$$v \longmapsto \int f v$$

take extension to $\overline{C^1(\bar{I})}^{H^1} = H^1$ ← density thm., so get $\ell \in (H^1)^*$ thus by Riesz $\exists!$ $u \in H^1$ s.t.

$\ell(v) = (u, v)_{H^1} \quad \forall\ v \in H^1$

To prove that such $u \in H^2$ is a classical sol. to (#) argue as above

$$(u')' = \underbrace{u - f}_{\in AC(\bar{I}) \hookrightarrow L^2(I)} \rightsquigarrow u' \in H^1$$

thus (fund. thm.) $u' \in AC(\bar{I})$ and we can represent it as

$$u'(x) = u'(x_0) + \int_{x_0}^{x} (u - f)(t)\,dt \qquad \rightsquigarrow u' \in C^1(\bar{I})$$
$$\rightsquigarrow u \in C^2(\bar{I})$$

How about boundary conditions?

Take $(\#)_w$ and integrate by parts (in a classical sense, since $u \in C^2(\bar{I})$ and wlog $v \in C^1(\bar{I})$)

$$0 = \int_a^b (-u'' + u - f)\, v\, dx + \Big[u'(b)v(b) - u'(a)v(a)\Big]$$

$$\Rightarrow \begin{cases} 0 = \int_a^b (-u'' + u - f)\, v \\ 0 = \big[u'(b)v(b) - u'(a)v(b)\big] \end{cases} \qquad \forall\, v \in C^1(\bar{I})$$

$$\Rightarrow \begin{cases} -u'' + u - f = 0 \\ u'(a) = u'(b) = 0 \end{cases}$$

■

After this worm up, we'll now consider

$f \in C^0(\bar{I})$, want to solve for $u \in C^2(\bar{I})$ the pb.

$$(\$) \qquad \begin{cases} -u'' = f & I \\ u'(a) = u'(b) = 0 & \partial I \end{cases}$$

Rmk.: i) necessary condition $\int_I f = 0$

(always assumed in the sequel)

ii) when solvable the equation has the following property:

if u is a sol. then $u + c$ is also a sol. $\forall\, c \in \mathbb{R}$

We need to set up the problem so to "kill" this invariance!

$$X = \{ u \in H^1(I) : \overline{u} := \fint_I u(x)\,dx = 0 \}$$

$X \subset H^1(I)$ closed linear subspace

Lemma: (like for H^1_0) the space X also enjoys a Poincaré (-Wirtinger) inequality: $\exists\, C = C(I)$

$$\|u\|_{L^2(I)} \le C \|u'\|_{L^2(I)} \qquad (\forall u \in X).$$

Pf: $|u(x)| = \left| \frac{1}{|I|} \int_I (u(x) - u(y))\,dy \right|$

$$\le \frac{1}{|I|} \int_I |u(x) - u(y)|\,dy \le \frac{1}{|I|} \cdot \|u'\|_{L^1} \cdot |I|$$

$$u(x) - u(y) = \int_x^y u'(t)\,dt$$

$$\Rightarrow |u(x) - u(y)| \le \int_x^y |u'(t)|\,dt \le \|u'\|_{L^1}$$

So $\|u\|_{L^2}^2 = \int_I |u(x)|^2 dx \le \|u'\|_{L^1}^2 |I|$

ineq. above $\le \|u'\|_{L^2}^2 \cdot |I|^2$ (Hölder)

thus proof is complete w/ $C = |I|^2$ ∎

Thanks to that inequality (exactly like we did for H^1_0) we can consider on $X \subset H^1$ the equivalent norm

$$\|u\|_X := \left(\int_I |u'|^2 \right)^{1/2}$$

which is induced by $(u, v)_X = \int_I u' \cdot v' \qquad (u, v \in X)$

Now back to Neumann problem ($\$$): natural notion of weak sol.

$$(\$)_w \qquad \int_I u' \cdot v' = \int_I f v \qquad (\forall v \in H^1(I))$$

Here we need a key observation:

Rem. if $\int_I f = 0$ then $(\$)_w$ holds if and only if

$(\$)_{vw}$ $\int_I u' \cdot v' = \int_I f v \quad (\forall v \in X)$

Thm. $\exists$ weak sol. $u \in X$ of $(\$)$, and in fact $u \in C^2(\bar{I})$ and is a classical solution.

Pf: weak sol: $\ell : X \longrightarrow \mathbb{R}$, $v \longmapsto \int_I f v$

check (Hölder) $\ell \in X^*$ so again by Riesz

$\exists!\ u \in X$ s.t. $\ell(v) = (u, v)_X \quad \forall v \in X.$

$\Leftrightarrow$ $\int_I u' v' = \int_I f v \quad (\forall v \in X)$

by rmk. $\int_I f = 0$ $\rightsquigarrow$ u is actually a weak sol. $\int_I u'v' = \int_I fv \quad \forall v \in H^1$

classical sol: says $\begin{cases} u' \in L^2 \\ u'' \in L^2 \end{cases}$ as above $\rightsquigarrow u \in C^2(\bar{I})$

then take and integrate by parts:

$$\int_I (-u'' - f) v = u'(b) v(b) - u'(a) v(a)$$

$$\forall v \in C^1(\bar{I})$$

$\Longrightarrow$ $LHS = RHS = 0$

i.e. $LHS = 0 \Longrightarrow u'' = -f$

$RHS = 0 \Longrightarrow u'(a) = u'(b) = 0$ ∎

Rem. we'll now build the tools to perform a similar roadmap for partial differential equations.

Sobolev spaces in $\mathbb{R}^N$

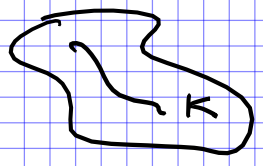
Key difference: if $u \in W^{1,p}(I)$ then $u \in AC$, and the weak derivative coincides a.e. with the pointwise derivative.

Fact 1: in $\mathbb{R}^N$, $N > 1$ there are lots of $W^{1,p}$ functions that are singular (i.e. pot. discontinuous)

Fact 2: we can still give a good criterion for the weak derivative to exist and be represented by pointwise der.

Suggestion: first work on problem 2.2 (ice-cream cone)

Criterion



$\Omega \subset \mathbb{R}^n$ open

$K \subset\subset \Omega$ "small" i.e. pot. $\mathcal{L}^n(K) = 0$

$u \in C^1(\Omega \setminus K)$, when is $\nabla u(x)$ ~ pointwise defined and C^0 on $\Omega \setminus K$

actually the weak gradient of u?

Smallness: want K to have null $W^{1,p}$ capacity i.e.

$K \subset\subset \mathbb{R}^n$, $\mathcal{L}^n(K) = 0$ w/ $\exists (\psi_k)$ in $C_c^\infty(\mathbb{R}^n)$ s.t.

- $\psi_k = 1$ on K (or a neighborhood of K)
- $\psi_k \to 0$ $\mathcal{L}^n$ a.e.
- $\| \nabla \psi_k \|_{L^p} \to 0$ $(k \to \infty)$

Rmk: by Poincaré $\| \nabla \psi_k \|_{L^p} \to 0$

$\Rightarrow \| \psi_k \|_{L^p} \to 0$

(implies $\mathcal{L}^n(K) = 0$)

Example: points have null $W^{1,p}$ capacity for $\boxed{1 \le p \le n}$.

$K = \{0\} \subset \mathbb{R}^n$, $n > 1$.

$\boxed{1 \le p < n}$ take any $\psi \in C_c^\infty(B_1(0))$, $\psi \ge 0$

$\psi(0) = 1$

set $\psi_k(x) := \psi(kx) \in C_c^\infty(B_{1/k}(0))$

$$\int_{B_1(0)} |\nabla \psi_k|^p = k^{p-n} \int_{B_1(0)} |\nabla \psi|^p \quad \rightsquigarrow \text{if } p < n$$

$\uparrow$ change of variable $y = kx$

$\longrightarrow 0 \quad (k \to \infty)$

$\boxed{p = n}$ still true, but need a more clever proof.

Can take ψ_k to be a radially symm function $\mathbb{R}^n \to \mathbb{R}$ which is a smoothing of

$$\vartheta_k(x) = \begin{cases} 1 & |x| \le \frac{1}{e^{e^{k+1}}} =: r_k \\ 0 & |x| \ge \frac{1}{e^{e^k}} = R_k \\ \log\log\left(\frac{1}{|x|}\right) - k & \text{in between} \end{cases}$$

$\vartheta_k \in W^{1,n}(B_1(0))$ and $\int_{B_1(0)} |\nabla \vartheta_k|^n \longrightarrow 0 \quad (k \to \infty)$

Then, to conclude the argument, may employ the 1D density theorem $C_c^\infty(\mathbb{R})$ dense $W^{1,p}$ $(1 \le p < \infty)$

Thm. $K \subset\subset \Omega \subset \mathbb{R}^n$, say $\mathcal{L}^n(\Omega) < \infty$.

$u \in C^1(\Omega \setminus K) \cap L^q(\Omega)$

$|\nabla u| \in L^p(\Omega \setminus K)$ $\quad 1 \le p \le q \le \infty$

If $\mathrm{Cap}_{W^{1,s}}(K) = 0$ for $\frac{1}{q} + \frac{1}{s} = 1$ then $u \in W^{1,p}(\Omega)$ and has weak derivative ∇u.

Pf: given K, let (ψ_k) be one of the sequences of cut-off functions for K and pick $\varphi \in C_c^\infty(\Omega)$ a test function.

Roughly speaking: want to "cut away" the region around K i.e. work with $\varphi(1-\psi_k)$ instead of φ.

$$\int_\Omega u \nabla \varphi = \lim_{k\to\infty} \int_\Omega u\,(\nabla(\varphi(1-\psi_k)))$$

$$\left|\int_\Omega u\varphi\nabla\psi_k\right| \le \|\varphi\|_{L^\infty}\|u\|_{L^q}\|\nabla\psi_k\|_{L^s} \longrightarrow 0 \quad (k\to\infty)$$

classical integration by parts

$$\left|\int_\Omega u\psi_k\nabla\varphi\right| \longrightarrow 0 \quad (k\to\infty)$$

dominated convergence

$$= -\lim_{k\to\infty}\int_\Omega (\nabla u)\cdot\varphi(1-\psi_k)$$

$$\left|\int_\Omega \nabla u\cdot\varphi\cdot\psi_k\right| \longrightarrow 0 \quad (k\to\infty)$$

dominated convergence

$$= -\int_\Omega (\nabla u)\cdot\varphi$$

$\rightsquigarrow$ u has weak gradient equal to the (pointwise defined) $\nabla u \in L^p(\Omega)$ ∎

Examples wait for next Monday.