1.1. The Dirichlet energy

(a) Apply integration by parts and Hölder's inequality.

(b) Use (a).

(c) Prove that the boundary of every connected component of Ω is a non-empty subset of the boundary of Ω.

1.2. The p-energy

(a) Exploit that the mapping $\mathbb{R}^n \ni x \mapsto |x|^p$ is strictly convex.

(b) Given any $\varphi \in C^2(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = 0$, compute $\frac{d}{dt}\big|_{t=0} E_p(u + t\varphi)$.

(c) About the notation: Denoting $\frac{\partial u}{\partial x_j} =: u_j$ and $\frac{\partial^2 u}{\partial x_j \partial x_k} =: u_{jk}$, we have

$$|\nabla u| = \Big(\sum_{j=1}^n u_j^2\Big)^{\frac{1}{2}}, \qquad D^2 u = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix}, \qquad |D^2 u| = \Big(\sum_{j=1}^n \sum_{k=1}^n u_{jk}^2\Big)^{\frac{1}{2}}.$$

Express the integrand as $|\nabla u|^p = |\nabla u|^{p-2} \nabla u \cdot \nabla u$ and integrate by parts. Prove and use the inequality $(\Delta u)^2 \leq n|D^2 u|^2$. Then apply Hölder's inequality in the form

$$\int_\Omega fgh\,dx \leq \Big(\int_\Omega |f|^p\,dx\Big)^{\frac{1}{p}} \Big(\int_\Omega |g|^q\,dx\Big)^{\frac{1}{q}} \Big(\int_\Omega |h|^r\,dx\Big)^{\frac{1}{r}}, \qquad 1 = \tfrac{1}{p} + \tfrac{1}{q} + \tfrac{1}{r}.$$

1.3. Laplace's equation

(a) Find and solve the ordinary differential equations for $v(x)$ and $w(y)$.

(b) Consider non-constant boundary data which are constant along two sides.

1.4. Mean-value property

(a) For $y \in \Omega$ and suitable $R > 0$, consider the function $\phi\colon\,]0, R[\, \to \mathbb{R}$ given by

$$\phi(r) = \fint_{\partial B_r(y)} u\,d\sigma = \fint_{\partial B_1(0)} u(y + rz)\,d\sigma(z).$$

Argue that $\phi(r)$ must be constant in r, compute $\phi'(r)$ and conclude that u is harmonic.

(b) Prove that $\phi(r)$ is constant if u is harmonic. Conclude

$$u(y) = \fint_{\partial B_r(y)} u \, d\sigma$$

and show how this implies

$$u(y) = \fint_{B_r(y)} u \, dx.$$

1.5. Liouville's theorem

(a) Exploit the mean-value property in balls of large radius.

(b) Compare the values of u at two different points using the mean-value property.

1.6. Harnack's inequality

Cover Q with balls of sufficiently small radius and exploit the mean-value property.