1.1. The Dirichlet energy

(a) Since $u \in C^2(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$ we may integrate by parts with vanishing boundary terms:

$$\int_\Omega |\nabla u|^2\,dx = -\int_\Omega u\Delta u\,dx \le \int_\Omega |u||\Delta u|\,dx \le \Big(\int_\Omega u^2\,dx\Big)^{\frac{1}{2}}\Big(\int_\Omega (\Delta u)^2\,dx\Big)^{\frac{1}{2}}.$$

The last estimate is Cauchy-Schwarz inequality.

(b) Fist, notice that any proper and open subset A of $\mathbb{R}^n$ has non vanishing boundary. Indeed, by contradiction, assume $\partial A = \emptyset$. Then $\overline{A} = \mathring{A} = A$, which means that A is both open and closed. But since $\mathbb{R}^n$ is connected, this contradicts that fact that A is proper. Since Ω is non-empty and bounded, then Ω is proper. By what we have shown so far, this leads to $\partial\Omega \neq \emptyset$. If $u \in C^2(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$ satisfies $\Delta u = 0$ in Ω, then

$$\int_\Omega |\nabla u|^2\,dx = -\int_\Omega u\Delta u\,dx = 0.$$

Since $|\nabla u(x)|^2 \ge 0$ for every $x \in \Omega$ we conclude $|\nabla u|^2 = 0$ in Ω. Since Ω is connected, then u is constant in Ω. By continuity, this constant must agree with the value of u on $\partial\Omega$; hence $u \equiv 0$.

(c) Yes, both statements still hold. Indeed, in (a) we have never used the connectedness of Ω. For what concerns (b), we still obtain $\nabla u = 0$ in Ω, which implies that u is constant on every connected component of Ω.
Let $\Omega' \subset \Omega$ be any connected component of Ω. Clearly, Ω' is non-empty and bounded. Since $\mathbb{R}^n$ is locally connected, Ω' is also open and thus $\partial\Omega' \neq \emptyset$, following the same argument ad above. Moreover, it follows easily that $\partial\Omega' \subset \partial\Omega$. Then, u is constant in Ω', $\partial\Omega' \neq \emptyset$ and $u|_{\partial\Omega'} = 0$. By continuity, this implies $u = 0$ in Ω'. By arbitrariness of Ω', the statement follows.

1.2. The *p*-energy

Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be open, bounded and regular, $2 \le p < \infty$ and $g \in C^2(\partial\Omega)$. Consider

$$E_p(u) := \int_\Omega |\nabla u|^p\,dx, \qquad\qquad \mathfrak{A} := \{u \in C^2(\overline{\Omega}) \mid u|_{\partial\Omega} = g\}.$$

(a) Suppose $u_1, u_2 \in \mathfrak{A}$ both satisfy

$$E_p(u_1) = E_p(u_2) = m =: \inf_{v\in\mathfrak{A}} E_p(v).$$

Since for $p \ge 2$ the mapping $\mathbb{R}^n \ni v \mapsto |v|^p$ is strictly convex, we have

$$\Big|\frac{v_1+v_2}{2}\Big|^p < \frac{|v_1|^p + |v_2|^p}{2}$$

for every $v_1, v_2 \in \mathbb{R}^n$ with $v_1 \neq v_2$. If $\nabla u_1 \neq \nabla u_2$ in a set of positive measure, then

$$E_p\Big(\frac{u_1+u_2}{2}\Big) = \int_\Omega \Big|\frac{\nabla u_1 + \nabla u_2}{2}\Big|^p \, dx < \int_\Omega \frac{|\nabla u_1|^p + |\nabla u_2|^p}{2} \, dx = m,$$

which is a contradiction to u_1 and u_2 being minimisers of E_p.
Consequently, $\nabla u_1 = \nabla u_2$ a.e. on Ω. Then, by continuity, $\nabla u_1 = \nabla u_2$ on Ω, which means that $u_1 - u_2$ is constant in every connected component of Ω. Since $(u_1-u_2)|_{\partial\Omega} = 0$, by the same argument that we have used in the previous exercise we conclude $u_1 = u_2$ in Ω.

(b) Suppose, $u \in \mathfrak{A}$ is a minimiser of E_p. Let $\varphi \in C^2(\overline{\Omega})$ satisfy $\varphi|_{\partial\Omega} = 0$. Then $u + t\varphi \in \mathfrak{A}$ for every $t \in \mathbb{R}$. Moreover,

$$\frac{d}{dt}\int_\Omega |\nabla u + t\nabla\varphi|^p \, dx = p\int_\Omega |\nabla u + t\nabla\varphi|^{p-2}(\nabla u + t\nabla\varphi)\cdot\nabla\varphi \, dx$$

In particular,

$$0 = \frac{d}{dt}\Big|_{t=0} E_p(u+t\varphi) = p\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi \, dx = -p\int_\Omega \operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big)\varphi \, dx$$

for every $\varphi \in C^2(\overline{\Omega})$ with $\varphi|_{\partial\Omega} = 0$. Hence, by the fundamental lemma of calculus of variations, $-\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = 0$ a.e. in Ω. By continuity, $-\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = 0$ in Ω.

(c) For every $u \in C^2(\overline{\Omega})$ with $u|_{\partial\Omega} = 0$ there holds

$$\begin{aligned}
\int_\Omega |\nabla u|^p \, dx &= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla u \, dx = -\int_\Omega \operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big)u \, dx \\
&= -\int_\Omega \Big((p-2)|\nabla u|^{p-4}\Big(D^2u(\nabla u,\nabla u)\Big) + |\nabla u|^{p-2}\Delta u\Big)u \, dx \\
&\leq \Big(p-2+\sqrt{n}\Big)\int_\Omega |\nabla u|^{p-2}|D^2u||u| \, dx,
\end{aligned}$$

where $(\Delta u)^2 \leq n|D^2u|^2$ is used. Indeed, with $\frac{\partial u}{\partial x_j} =: u_j$ and $\frac{\partial^2 u}{\partial x_j \partial x_k} =: u_{jk}$, we have

$$\begin{aligned}
\Big|D^2u(\nabla u,\nabla u)\Big| &= \Big|\sum_{j=1}^n u_j \sum_{k=1}^n u_{jk}u_k\Big| \leq \Big(\sum_{j=1}^n u_j^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^n\Big(\sum_{k=1}^n u_{jk}u_k\Big)^2\Big)^{\frac{1}{2}} \\
&\leq |\nabla u|\Big(\sum_{j=1}^n\Big(\sum_{k=1}^n u_{jk}^2\Big)\Big(\sum_{k=1}^n u_k^2\Big)\Big)^{\frac{1}{2}} = |\nabla u|^2\Big(\sum_{j=1}^n\sum_{k=1}^n u_{jk}^2\Big)^{\frac{1}{2}} = |\nabla u|^2|D^2u|, \\
\Big(\frac{\Delta u}{n}\Big)^2 &= \Big(\frac{u_{11}+\ldots+u_{nn}}{n}\Big)^2 \leq \frac{u_{11}^2+\ldots+u_{nn}^2}{n} \leq \frac{1}{n}\sum_{j=1}^n\sum_{k=1}^n u_{jk}^2 = \frac{1}{n}|D^2u|^2.
\end{aligned}$$

Applying Hölder's inequality with $1 = \frac{p-2}{p} + \frac{1}{p} + \frac{1}{p}$, we obtain

$$\begin{aligned}
&\int_\Omega |\nabla u|^p \, dx \leq \left(p - 2 + \sqrt{n}\right) \left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{p-2}{p}} \left(\int_\Omega |D^2 u|^p \, dx\right)^{\frac{1}{p}} \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}}, \\
\Rightarrow &\left(\int_\Omega |\nabla u|^p \, dx\right)^{\frac{2}{p}} \leq \left(p - 2 + \sqrt{n}\right) \left(\int_\Omega |D^2 u|^p \, dx\right)^{\frac{1}{p}} \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{p}}, \\
\Rightarrow \quad &\int_\Omega |\nabla u|^p \, dx \leq \left(p - 2 + \sqrt{n}\right)^{\frac{p}{2}} \left(\int_\Omega |D^2 u|^p \, dx\right)^{\frac{1}{2}} \left(\int_\Omega |u|^p \, dx\right)^{\frac{1}{2}}.
\end{aligned}$$

1.3. Laplace's equation

(a) If $u \in C^2(\Omega)$ is of the form $u(x,y) = v(x)w(y)$, then

$$(\Delta u)(x,y) = v''(x)\, w(y) + v(x)\, w''(y).$$

Assume that $\Delta u = 0$. Define

$$I := \{x \in]a,b[\text{ s.t. } v(x) \neq 0\}$$

and

$$J := \{y \in]c,d[\text{ s.t. } w(y) \neq 0\}.$$

Since I and J are both open (by continuity of v and w), then $Q := I \times J$ is an open subset of Ω. Then, at every $(x,y) \in Q$ we obtain

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)}. \tag{‡}$$

Since the left hand side depends only on x and the right hand side only on y, the equation requires both sides to be constant. More precisely,

$$\frac{v''(x)}{v(x)} = \kappa = -\frac{w''(y)}{w(y)}$$

at every $(x,y) \in \Omega$, where $v(x)w(y) \neq 0$. The resulting equations

$$v''(x) = \kappa v(x), \qquad\qquad w''(y) = -\kappa w(y)$$

can be solved separately by distinguishing three cases.

Case 1. $\kappa = \lambda^2$ for some $\lambda > 0$. Then, with constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$

$$v(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x}, \qquad w(y) = C_3 \sin(\lambda y) + C_4 \cos(\lambda y).$$

Case 2. $\kappa = 0$. Then, with constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$

$$v(x) = C_1 x + C_2, \qquad w(y) = C_3 y + C_4.$$

Case 3. $\kappa = -\lambda^2$ for some $\lambda > 0$. Then, with constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$

$$v(x) = C_1 \sin(\lambda x) + C_2 \cos(\lambda x), \qquad w(y) = C_3 e^{\lambda y} + C_4 e^{-\lambda y}.$$

Clearly, all the functions that we have found so far actually belong to $C^2(\Omega)$. Moreover, a direct computation shows that they are all harmonic in all of Ω, not only on Q. Since we know that $u|_Q$ coincides with the restriction to Q of one of these functions, the statement follows directly by the unique continuation principle.

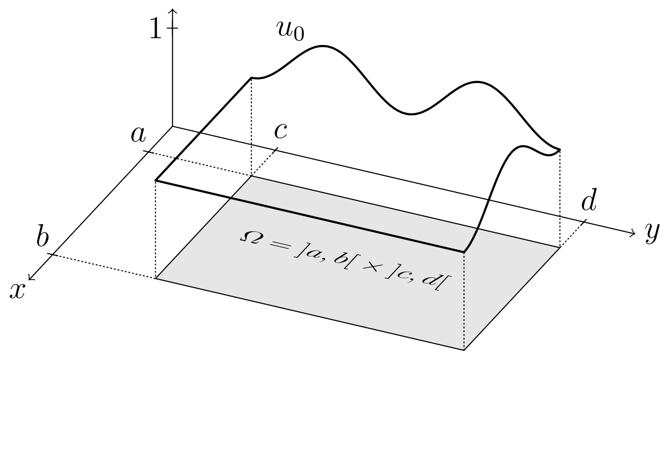
(b) Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$ and let $\Omega :=]a, b[\times]c, d[\subset \mathbb{R}^2$. Let $u_0 \in C^2(\partial\Omega)$ be non-constant satisfying

$$\forall x \in [a, b] \quad u_0(x, c) = 1, \qquad \forall y \in [c, d] \quad u_0(b, y) = 1.$$

Then, any function $u(x, y) = v(x)w(y)$ in Ω with $u|_{\partial\Omega} = u_0$ must satisfy

$$\begin{aligned} \forall x \in [a, b] \quad 1 = u_0(x, c) = u(x, c) = v(x)w(c) \quad &\Rightarrow v(x) = \tfrac{1}{w(c)}, \\ \forall y \in [c, d] \quad 1 = u_0(b, y) = u(b, y) = v(b)w(y) \quad &\Rightarrow w(y) = \tfrac{1}{v(b)}. \end{aligned}$$

In particular, both v and w must be constant. This however is in contradiction to u_0 being non-constant.



1.4. Mean-value property

(a) Let $\Omega \subset \mathbb{R}^n$ be open. Let $y \in \Omega$ and $R > 0$ such that such that $B_R(y) \subset \Omega$. Given $u \in C^2(\Omega)$, we define $\phi\colon]0, R[\to \mathbb{R}$ by

$$\phi(r) = \fint_{\partial B_r(y)} u \, d\sigma = \fint_{\partial B_1(0)} u(y + rz) \, d\sigma(z)$$

and compute

$$\begin{aligned}\phi'(r) &= \fint_{\partial B_1(0)} \frac{d}{dr}\Big(u(y + rz)\Big) \, d\sigma(z) = \fint_{\partial B_1(0)} z \cdot \nabla u(y + rz) \, d\sigma(z) \\ &= \fint_{\partial B_r(y)} \frac{\xi - y}{r} \cdot \nabla u(\xi) \, d\sigma(\xi) = \frac{r}{n} \fint_{B_r(y)} \Delta u \, dx, \end{aligned} \tag{†}$$

where the divergence theorem applies because $\nu = \frac{\xi - y}{r}$ is the outward unit normal vector along $\partial B_r(y)$. If u satisfies the mean-value property, ϕ is constant. In particular,

$$0 = \phi'(r) = \frac{r}{n} \fint_{B_r(y)} \Delta u \, dx. \tag{$*$}$$

By assumption, Δu is continuous. If $\Delta u \neq 0$, there exist $y \in \Omega$ and $r > 0$ such that either $\Delta u < 0$ in $B_r(y)$ or $\Delta u > 0$ in $B_r(y)$ which contradicts $(*)$ in both cases.

(b) Let $u \in C^2(\Omega)$ be harmonic. As in (a) let $y \in \Omega$ and $R > 0$ such that $B_R(y) \subset \Omega$. Since $\Delta u = 0$, equation (†) in part (a) yields

$$\phi'(r) = \frac{r}{n} \fint_{B_r(y)} \Delta u \, dx = 0 \tag{1}$$

which implies that the map $\phi\colon]0, R[\to \mathbb{R}$ given by

$$\phi(r) = \fint_{\partial B_r(y)} u \, d\sigma$$

is constant in r. In particular,

$$\fint_{\partial B_r(y)} u \, d\sigma = \lim_{r \to 0} \fint_{\partial B_r(y)} u \, d\sigma = u(y)$$

which proves the first part of the mean-value property. Moreover,

$$\begin{aligned}\fint_{B_r(y)} u \, dx &= \frac{1}{|B_r|} \int_0^r \Big(\int_{\partial B_\rho(y)} u \, d\sigma \Big) \, d\rho = \frac{1}{|B_r|} \int_0^r |\partial B_\rho| \Big(\fint_{\partial B_\rho(y)} u \, d\sigma \Big) \, d\rho \\ &= \frac{u(y)}{|B_r|} \int_0^r |\partial B_\rho| \, d\rho = u(y)\end{aligned}$$

which proves the second part of the mean-value property.

1.5. Liouville's theorem

(a) Let $u \in C^2(\mathbb{R}^n)$ be harmonic and $u \in L^1(\mathbb{R}^n)$. Let $B_r(y) \subset \mathbb{R}^n$ be the open ball of radius $r > 0$ around y. The mean-value property proven in problem 1.4 (b) implies

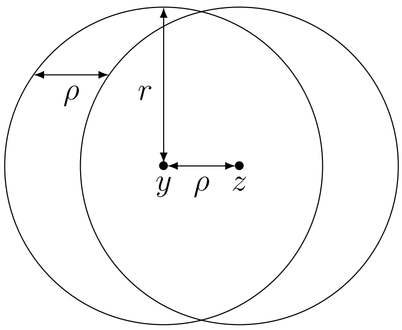
$$|u(y)| = \left| \fint_{B_r(y)} u\,dx \right| \leq \frac{1}{|B_r|} \int_{B_r(y)} |u|\,dx \leq \frac{1}{|B_r|} \|u\|_{L^1(\mathbb{R}^n)} \xrightarrow{r\to\infty} 0.$$

Since $y \in \mathbb{R}^n$ is arbitrary, we obtain $u \equiv 0$.

(b) Let $u \in C^2(\mathbb{R}^n)$ be harmonic and $|u| \leq c_0$. Let $y, z \in \mathbb{R}^n$ be two arbitrary points and $\rho := |y - z|$. Then, for every $r > \rho$, the mean-value property implies

$$\begin{aligned} u(y) - u(z) &= \fint_{B_r(y)} u\,dx - \fint_{B_r(z)} u\,dx \\ &= \frac{1}{|B_r|} \int_{B_r(y)\setminus B_r(z)} u\,dx - \frac{1}{|B_r|} \int_{B_r(z)\setminus B_r(y)} u\,dx \\ &\leq \frac{2c_0}{|B_r|} |B_r(y) \setminus B_r(z)| \leq \frac{2c_0\,\rho\,|B_r^{\mathbb{R}^{n-1}}|}{|B_r^{\mathbb{R}^n}|} \xrightarrow{r\to\infty} 0 \end{aligned}$$

i. e. $u(y) \leq u(z)$. By switching the roles of y and z we also obtain $u(z) \leq u(y)$, i. e. $u(y) = u(z)$. Since $y, z \in \mathbb{R}^n$ are arbitrary, we conclude that u is constant.



1.6. Harnack's inequality

Given the open set $\Omega \subset \mathbb{R}^n$ and the connected open subset $Q \subset \Omega$ such that $\overline{Q} \subset \Omega$, let $r = \frac{1}{4}\operatorname{dist}(Q, \partial\Omega) > 0$. Let $u \in C^2(\Omega)$ be harmonic. According to the mean-value property proven in problem 1.4 (b) and since u is non-negative,

$$u(y) = \frac{1}{|B_{2r}|} \int_{B_{2r}(y)} u\,dx \geq \frac{1}{|B_{2r}|} \int_{B_r(z)} u\,dx = \frac{1}{2^n |B_r|} \int_{B_r(z)} u\,dx = \frac{1}{2^n} u(z)$$

for any $y, z \in Q$ with $|z - y| < r$. Since $\overline{Q}$ is connected and compact, there exist finitely many $x_1, \ldots, x_m \in Q$ such that $Q \subset \bigcup_{i=1}^m B_r(x_i)$ and such that $|x_i - x_{i+1}| < r$ for $i = 2, \ldots, m$. Consequently,

$$\forall x, y \in Q \quad u(x) \geq 2^{-n(m+1)} u(y) \qquad\qquad \Rightarrow \sup_Q u \leq 2^{n(m+1)} \inf_Q u.$$