11.1. Wrapping up Lecture 21

1. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be linear, symmetric and invertible. Given $v \in H^1(B_r^+)$, let $w = v \circ T^{-1}$. Then, using symmetry of T,

$$\begin{aligned}\frac{\partial w}{\partial \nu} &= \nu \cdot \nabla w = (Te_n) \cdot \nabla(v \circ T^{-1}) = (Te_n) \cdot \Big(T^{-1}((\nabla v) \circ T^{-1})\Big) \\ &= e_n \cdot \Big(TT^{-1}((\nabla v) \circ T^{-1})\Big) = e_n \cdot ((\nabla v) \circ T^{-1}) = \frac{\partial v}{\partial x_n} \circ T^{-1}.\end{aligned} \tag{1}$$

By using the change of variables $x = T^{-1}y$, we eventually get

$$\begin{aligned}\int_{B_r^+} \Big|\frac{\partial v}{\partial x_n}(x)\Big|^2 dx &= \int_{T(B_r^+)} \Big|\frac{\partial v}{\partial x_n}(T^{-1}y)\Big|^2 |\det(T^{-1})|\, dy \\ &= |\det(T^{-1})| \int_{T(B_r^+)} \Big|\frac{\partial w}{\partial \nu}(y)\Big|^2 dy.\end{aligned}$$

2. Let $\bar v$ be the odd reflection of v with respect to $\partial\mathbb{R}^n_+ \cap B_R$ and define

$$\bar w := T \circ \bar v. \tag{2}$$

One can easily check that $\bar w$ is harmonic on $T(B_R)$. Since $\bar w$ is harmonic on the ellipsoid $T(B_R)$ and ν is a constant direction, we get that $\partial \bar w/\partial\nu$ is harmonic on $T(B_R)$. Set

$$u := \frac{\partial \bar w}{\partial \nu} - \overline{\left(\frac{\partial \bar v}{\partial x_n}\right)}_{B_R}$$

and notice that $u \in H^1_{loc}(T(B_R))$ is harmonic. Fix $\delta = \delta(A^{(0)}) \in (0,1)$ such that $B_\delta \subset T(B_1) \subset B_{1/\delta}$. By Poincaré-Wirtinger inequality, part (i) of Lemma 1 in lecture 21 and the variance-minimizing property of the mean-value, we get that for every $0 < r < \delta^2 R/2$ we can estimate

$$\begin{aligned}\int_{T(B_r)} \left|\frac{\partial \bar w}{\partial \nu} - \overline{\left(\frac{\partial \bar w}{\partial \nu}\right)}_{T(B_r)}\right|^2 dx &= \int_{T(B_r)} \left|u - \overline{(u)}_{T(B_r)}\right|^2 dx = \min_{a\in\mathbb{R}} \int_{T(B_r)} |u-a|^2\, dx \\ &\le \int_{T(B_r)} \left|u - \overline{(u)}_{B_{r/\delta}}\right|^2 dx \le \int_{B_{r/\delta}} \left|u - \overline{(u)}_{B_{r/\delta}}\right|^2 dx \\ &\le C\left(\frac{r}{\delta}\right)^2 \int_{B_{r/\delta}} |\nabla u|^2\, dx \\ &\le C\left(\frac{r}{\delta}\right)^2 \left(\frac{2r}{\delta^2 R}\right)^n \int_{B_{\delta R/2}} |\nabla u|^2\, dx\end{aligned}$$

$$
\begin{aligned}
&\leq C\left(\frac{r}{R}\right)^{n+2}\int_{B_{\delta R}}|u|^2\,dx\\
&= C\left(\frac{r}{R}\right)^{n+2}\int_{B_{\delta R}}\left|\frac{\partial\bar{w}}{\partial\nu}-\overline{\left(\frac{\partial\bar{v}}{\partial x_n}\right)}_{B_R}\right|^2 dx\\
&\leq C\left(\frac{r}{R}\right)^{n+2}\int_{T(B_R)}\left|\frac{\partial\bar{w}}{\partial\nu}-\overline{\left(\frac{\partial\bar{v}}{\partial x_n}\right)}_{B_R}\right|^2 dx,
\end{aligned}
$$

where $C=C(n,A^{(0)})>0$ is a generic constant depending just on n and $A^{(0)}$. By (2), we get

$$
\int_{B_r^+}\left|\frac{\partial v}{\partial x_n}-\overline{\left(\frac{\partial v}{\partial x_n}\right)}_{B_r^+}\right|^2 dx=\frac{1}{2}\int_{B_r}\left|\frac{\partial\bar{v}}{\partial x_n}-\overline{\left(\frac{\partial\bar{v}}{\partial x_n}\right)}_{B_r}\right|^2 dx, \tag{3}
$$

$$
\int_{B_R^+}\left|\frac{\partial v}{\partial x_n}-\overline{\left(\frac{\partial v}{\partial x_n}\right)}_{B_R^+}\right|^2 dx=\frac{1}{2}\int_{B_R}\left|\frac{\partial\bar{v}}{\partial x_n}-\overline{\left(\frac{\partial\bar{v}}{\partial x_n}\right)}_{B_R}\right|^2 dx, \tag{4}
$$

for every $0<r<R$. Moreover, by the same computation as in (1), we get

$$
\frac{\partial\bar{v}}{\partial x_n}=\frac{\partial\bar{w}}{\partial\nu}\circ T.
$$

Again, by the variance-minimizing property the mean-value we obtain

$$
\begin{aligned}
\int_{B_r}\left|\frac{\partial\bar{v}}{\partial x_n}-\overline{\left(\frac{\partial\bar{v}}{\partial x_n}\right)}_{B_r}\right|^2 dx &= \min_{a\in\mathbb{R}}\int_{B_r}\left|\frac{\partial\bar{v}}{\partial x_n}-a\right|^2 dx\\
&\leq \int_{B_r}\left|\frac{\partial\bar{v}}{\partial x_n}-\overline{\left(\frac{\partial\bar{w}}{\partial\nu}\right)}_{T(B_r)}\right|^2 dx\\
&= |\det(T^{-1})|\int_{T(B_r)}\left|\frac{\partial\bar{w}}{\partial\nu}-\overline{\left(\frac{\partial\bar{w}}{\partial\nu}\right)}_{T(B_r)}\right|^2 dx\\
&\leq |\det(T^{-1})|C\left(\frac{r}{R}\right)^{n+2}\int_{T(B_R)}\left|\frac{\partial\bar{w}}{\partial\nu}-\overline{\left(\frac{\partial v}{\partial x_n}\right)}_{B_R^+}\right|^2 dx,\\
&= C\left(\frac{r}{R}\right)^{n+2}\int_{B_R}\left|\frac{\partial\bar{v}}{\partial x_n}-\overline{\left(\frac{\partial v}{\partial x_n}\right)}_{B_R}\right|^2 dx,
\end{aligned}
$$

for every $0<r<\delta^2R/2$. By combining the previous estimate, (3) and (4), we conclude that

$$
\int_{B_r^+}\left|\frac{\partial\bar{v}}{\partial x_n}-\overline{\left(\frac{\partial\bar{v}}{\partial x_n}\right)}_{B_r^+}\right|^2 dx\leq C\left(\frac{r}{R}\right)^{n+2}\int_{B_R^+}\left|\frac{\partial\bar{v}}{\partial x_n}-\overline{\left(\frac{\partial v}{\partial x_n}\right)}_{B_R^+}\right|^2 dx,
$$

for every $0 < r < \delta^2 R/2$. Finally, notice that for $\delta^2 R/2 \leq r < R$ we have

$$\int_{B_r^+} \left| \frac{\partial \bar{v}}{\partial x_n} - \overline{\left(\frac{\partial \bar{v}}{\partial x_n}\right)}_{B_r^+} \right|^2 dx \leq \left(\frac{2}{\delta^2}\right)^{n+2} \left(\frac{r}{R}\right)^{n+2} \int_{B_R^+} \left| \frac{\partial \bar{v}}{\partial x_n} - \overline{\left(\frac{\partial v}{\partial x_n}\right)}_{B_R^+} \right|^2 dx.$$

Hence, the statement follows.

Let $A^{(0)}$ be a constant coefficients elliptic operator and let $v \in H^1(B_R^+)$ be a solution of

$$A^{(0)} v = 0 \tag{5}$$

on B_R^+ such that $\mathrm{tr}(v) = 0$ on $\partial\mathbb{R}^n_+ \cap B_R$. We may wonder why we have to go through the linear isomorphism T instead of simply working with the odd reflection of v, as we have done for $A^{(0)} = -\Delta$. The issue is that whenever the operator $A^{(0)}$ doesn't commute with odd reflection operator, the odd reflection of v may not be a solution of (5) on the whole ball B_R, as we can see by the following example.

Remark. Let $B_1 \subset \mathbb{R}^2$ be the open unit ball in $\mathbb{R}^2$ and consider the elliptic operator $A^{(0)}$ given by

$$A^{(0)} u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2}.$$

Clearly, the function $u \in C^\infty(B_1^+)$ given by $u(x,y) = 2xy - y^2$ solves the equation $A^{(0)} u = 0$ on B_1^+ and has vanishing trace on $\partial\mathbb{R}^2_+ \cap B_1$. Nevertheless, its odd reflection with respect to $\partial\mathbb{R}^2_+ \cap B_1$, i.e.

$$\bar{u}(x,y) = \begin{cases} 2xy - y^2 & \forall\, (x,y) \in B_1^+ \\ 2xy + y^2 & \forall\, (x,y) \in B_1 \setminus B_1^+ \end{cases}$$

doesn't solve the equation $A^{(0)} \bar{u} = 0$ on the whole ball B_1.

11.2. Divergence theorem reloaded

Let $w \in H^1_0(B_r)$. By definition of H^1_0, there exists a sequence $(w_k)_{k\in\mathbb{N}}$ in $C^\infty_c(B_r)$ such that $\|w - w_k\|_{H^1(B_r)} \to 0$ as $k \to \infty$. Moreover, integration by parts yields

$$\int_{B_r} \frac{\partial w_k}{\partial x_i}\, dx = \int_{\partial B_r} w_k \nu_i \, d\sigma, = 0,$$

where $\nu_i(x) := x_i/|x|$ on ∂B_r.
Therefore,

$$\left| \int_{B_r} \frac{\partial w}{\partial x_i}\, dx \right| = \left| \int_{B_r} \frac{\partial w}{\partial x_i} - \frac{\partial w_k}{\partial x_i}\, dx \right|$$

$$\leq \int_{B_r} |\nabla w - \nabla w_k| \, dx \leq |B_r|^{\frac{1}{2}} \|\nabla w - \nabla w_k\|_{L^2(B_r)} \xrightarrow{k\to\infty} 0.$$

Since the left hand side does not depend on k it must vanish as claimed.

11.3. Basic iteration lemma

First part: warm-up. Let $f\colon]0,R_0] \to [0,\infty[$ be a non-decreasing function satisfying

$$\forall \rho \in]0,R_0]: \quad f\Big(\frac{\rho}{2}\Big) \leq \Big(\frac{1}{2}\Big)^{\alpha} f(\rho)$$

for some $\alpha > 0$. Given $0 < r < R \leq R_0$, let $N \in \mathbb{N}$ such that $2^{-N-1}R < r \leq 2^{-N}R$. By monotonicity of f and iteration of the hypothesis, we obtain

$$f(r) \leq f\Big(\frac{R}{2^N}\Big) \leq \Big(\frac{1}{2}\Big)^{\alpha N} f(R) = \Big(\frac{2}{R}\Big)^{\alpha}\Big(\frac{R}{2^{N+1}}\Big)^{\alpha} f(R) < \Big(\frac{2r}{R}\Big)^{\alpha} f(R).$$

Second part: refinements. Let $f\colon]0,R_0] \to [0,\infty[$ be a non-decreasing function satisfying for some coefficients $A, B \geq 0$, some exponents $0 < \beta < \alpha$ and some $\varepsilon \geq 0$ the following inequality.

$$\forall 0 < r < R \leq R_0: \quad f(r) \leq A\Big(\Big(\frac{r}{R}\Big)^{\alpha} + \varepsilon\Big) f(R) + BR^{\beta}.$$

(a) For any $0 < R \leq R_0$ and any $\tau \in]0,1[$ we have by assumption

$$f(\tau R) \leq A(\tau^{\alpha} + \varepsilon) f(R) + BR^{\beta}.$$

Since increasing A weakens the assumption, we may assume $A > \frac{1}{2}$ without loss of generality. Suppose, $\varepsilon \leq (2A)^{-\frac{\alpha}{\alpha-\gamma}} =: \varepsilon_0$ for some $\gamma \in]\beta,\alpha[$. Note that since $\alpha > \gamma$ and $2A > 1$, there exists $\tau \in]0,1[$ such that $2A\tau^{\alpha} = \tau^{\gamma}$. In particular, $\varepsilon \leq (2A)^{-\frac{\alpha}{\alpha-\gamma}} = \tau^{\alpha}$ and $A(\tau^{\alpha} + \varepsilon) \leq 2A\tau^{\alpha} = \tau^{\gamma}$. Therefore, $f(\tau R) \leq \tau^{\gamma} f(R) + BR^{\beta}$ for any $0 < R \leq R_0$ as claimed.

(b) In part (a), we proved the claim for $k = 1$. Suppose, the inequality

$$f(\tau^k R) \leq \tau^{k\gamma} f(R) + BR^{\beta}\tau^{(k-1)\beta} \sum_{n=0}^{k-1} \tau^{n(\gamma-\beta)}$$

is true for some $k \in \mathbb{N}$. Then, by (a)

$$\begin{aligned} f(\tau^{k+1}R) &\leq \tau^{\gamma} f(\tau^k R) + B(\tau^k R)^{\beta} \\ &\leq \tau^{(k+1)\gamma} f(R) + \tau^{\gamma} BR^{\beta}\tau^{(k-1)\beta} \sum_{n=0}^{k-1} \tau^{n(\gamma-\beta)} + B(\tau^k R)^{\beta} \end{aligned}$$

$$
\begin{aligned}
&= \tau^{(k+1)\gamma} f(R) + BR^\beta \tau^{k\beta}\Big(\tau^{\gamma-\beta}\sum_{n=0}^{k-1}\tau^{n(\gamma-\beta)} + 1\Big)\\
&= \tau^{(k+1)\gamma} f(R) + BR^\beta \tau^{k\beta}\sum_{n=0}^{k}\tau^{n(\gamma-\beta)}
\end{aligned}
$$

and the claim follows by induction.

(c) Given $0 < r < R \leq R_0$ let $N \in \mathbb{N}_0$ such that $\tau^{N+1}R < r \leq \tau^N R$. Then, by (b)

$$
\begin{aligned}
f(r) \leq f(\tau^N R) &\leq \tau^{N\gamma} f(R) + BR^\beta \tau^{(N-1)\beta}\sum_{n=0}^{N-1}\tau^{n(\gamma-\beta)}\\
&\leq \tau^{-\gamma}\tau^{(N+1)\gamma} f(R) + BR^\beta\tau^{(N+1)\beta}\frac{\tau^{-2\beta}}{1-\tau^{(\gamma-\beta)}}\\
&\leq \tau^{-\gamma}\Big(\frac{r}{R}\Big)^\gamma f(R) + Br^\beta\frac{\tau^{-2\beta}}{1-\tau^{(\gamma-\beta)}}\\
&\leq C\Big(\Big(\frac{r}{R}\Big)^\beta f(R) + Br^\beta\Big),
\end{aligned}
$$

where $C := \max\Big\{\tau^{-\gamma}, \frac{\tau^{-2\beta}}{1-\tau^{(\gamma-\beta)}}\Big\}$ depends only on α, β, γ and A.

11.4. Interpolation inequality

Towards a contradiction, suppose there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in X and some $\varepsilon_0 > 0$ such that

$$\forall k \in \mathbb{N}: \quad 1 = \|x_k\|_Y \geq \varepsilon_0\|x_k\|_X + k\|x_k\|_Z.$$

Then, $\|x_k\|_X \leq \frac{1}{\varepsilon_0}$ and $\|x_k\|_Z \leq \frac{1}{k}$ for every $k \in \mathbb{N}$. Thus, the sequence $(x_k)_{k\in\mathbb{N}}$ is bounded in X and since the embedding $X \hookrightarrow Y$ is compact, there exists a subsequence $(x_k)_{k\in\Lambda\subset\mathbb{N}}$ and some $y \in Y$ such that $\|x_k - y\|_Y \to 0$ as $\Lambda \ni k \to \infty$. Since the embedding $Y \hookrightarrow Z$ is continuous, we also have $\|x_k - y\|_Z \to 0$ as $\Lambda \ni k \to \infty$. Consequently,

$$1 = \lim_{\Lambda\ni k\to\infty}\|x_k\|_Y = \|y\|_Y, \qquad\qquad \|y\|_Z = \lim_{\Lambda\ni k\to\infty}\|x_k\|_Z = 0$$

which is a contradiction.

11.5. Abstract method of continuity

Given $A_0, A_1 \in L(X,Y)$ let $A_t = (1-t)A_0 + tA_1$ for every $t \in [0,1]$ and assume that

$$\exists C < \infty \quad \forall t \in [0,1] \quad \forall x \in X: \quad \|x\|_X \leq C\|A_t x\|_Y. \tag{$*$}$$

The claim is equivalence of the statements

(i) A_0 is surjective.

(ii) A_1^* is injective with closed image.

(a) Let $I := \{t \in [0,1] : A_t \text{ is surjective}\}$. If we assume statement (i), then $0 \in I$. If we assume statement (ii), then $1 \in I$ by Satz 6.2.2. Therefore, $I \neq \emptyset$ in both cases.

(b) Let $t_0 \in I := \{t \in [0,1] : A_t \text{ is surjective}\}$. Assumption $(*)$ implies that A_{t_0} is also injective and that the inverse is continuous: $A_{t_0}^{-1} \in L(Y,X)$. For any $t \in [0,1]$, we have

$$A_t = A_{t_0} - (A_{t_0} - A_t) = \Big(1 - (A_{t_0} - A_t)A_{t_0}^{-1}\Big)A_{t_0},$$
$$A_{t_0} - A_t = (1-t_0)A_0 + t_0A_1 - (1-t)A_0 - tA_1 = (t-t_0)(A_0 - A_1).$$

Let $B := (A_{t_0} - A_t)A_{t_0}^{-1} \in L(Y,Y)$. By Satz 2.2.7 the operator $(1-B)$ is invertible with inverse $(1-B)^{-1} \in L(Y,Y)$ and in particular surjective, if $\|B\| < 1$. Since

$$\|B\| \leq \|A_{t_0} - A_t\|\|A_{t_0}^{-1}\| = |t-t_0|\|A_0 - A_1\|\|A_{t_0}^{-1}\|$$

we guarantee surjectivity of $(1-B)$ if $t \in [0,1]$ satisfies $|t-t_0| < \left(\|A_0 - A_1\|\|A_{t_0}^{-1}\|\right)^{-1}$. In this case we obtain that A_t is surjective, since A_{t_0} is surjective by assumption. Therefore, the set $I \subset [0,1]$ is open.

(c) Let $(t_k)_{k\in\mathbb{N}}$ be a sequence in I such that $t_k \to t_\infty$ as $k \to \infty$ for some $t_\infty \in [0,1]$. We claim that $A_{t_\infty} \in L(X,Y)$ is surjective. Let $y \in Y$ be arbitrary. Since $t_k \in I$, there exists $x_k \in X$ such that $A_{t_k}x_k = y$ for every $k \in \mathbb{N}$. Moreover, by assumption $(*)$,

$$\begin{aligned}
\|x_k - x_n\|_X &\leq C\|A_{t_k}(x_k - x_n)\|_Y \\
&= C\|A_{t_k}x_k - A_{t_n}x_n + (A_{t_n} - A_{t_k})x_n\|_Y \\
&= C\|(A_{t_n} - A_{t_k})x_n\|_Y \\
&\leq C\|A_{t_n} - A_{t_k}\|\|x_n\|_X \\
&\leq C^2|t_k - t_n|\|A_0 - A_1\|\|A_{t_n}x_n\|_Y = C^2|t_k - t_n|\|A_0 - A_1\|\|y\|_Y
\end{aligned}$$

which implies that $(x_k)_{k\in\mathbb{N}}$ is a Cauchy-sequence in X. Since $(X, \|\cdot\|_X)$ is complete, $(x_k)_{k\in\mathbb{N}}$ has a limit $x_\infty \in X$. Moreover,

$$\begin{aligned}
\|y - A_{t_\infty}x_\infty\|_Y &= \|A_{t_k}x_k - A_{t_\infty}x_\infty\|_Y \\
&= \|(A_{t_k} - A_{t_\infty})x_k + A_{t_\infty}(x_k - x_\infty)\|_Y \\
&\leq C\|A_{t_k} - A_{t_\infty}\|\|y\|_Y + \|A_{t_\infty}\|\|x_k - x_\infty\|_X \\
&\leq C|t_\infty - t_k|\|A_0 - A_1\|\|y\|_Y + \|A_{t_\infty}\|\|x_k - x_\infty\|_X \xrightarrow{k\to\infty} 0.
\end{aligned}$$

Hence, $A_{t_\infty}x_\infty = y$. Since $y \in Y$ is arbitrary, $t_\infty \in I$ follows. Therefore, the set $I \subset [0,1]$ is closed.

Since $[0,1]$ is a connected topological space and $I \subset [0,1]$ both open and closed by (b) and (c), we have either $I = \emptyset$ or $I = [0,1]$. According to Satz 6.2.2, A_1 is surjective if and only if A_1^* is injective with closed image. Hence, equivalence of (i) and (ii) follows:

$$
\begin{array}{ccccccccc}
\text{(i)} & \Leftrightarrow & 0 \in I & \Rightarrow & I = [0,1] & \Rightarrow & A_1 \text{ surjective} & \Leftrightarrow & \text{(ii)} \\
\text{(ii)} & \Leftrightarrow & 1 \in I & \Rightarrow & I = [0,1] & \Rightarrow & A_0 \text{ surjective} & \Leftrightarrow & \text{(i)}
\end{array}
$$