COMPLETE SOLUTION OF PROBLEM 12.13

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Achtung: this material is *not examinable* and is just provided for the sake of completeness.

We recall that in order to solve Problem 12.13 it's required to prove the following statement.

Proposition 1.1. *Let $B_1 \subset \mathbb{R}^2$ be the open unit ball in $\mathbb{R}^2$ and let $u \in C^{1,\alpha}(B_1)$ be a weak solution of the minimal surface equation*

$$\operatorname{div}\left(\frac{1}{\sqrt{1+|\nabla u|^2}}\nabla u\right) = 0. \tag{1.1}$$

Then, $u \in C^{2,\alpha}_{loc}(B_1)$.

Proof. Since $u \in C^{1,\alpha}(B_1)$, it holds that $||\nabla u||_{L^\infty(B_1)} < \infty$. Fix $M > 0$ such that $||\nabla u||_{L^\infty} < M$. Define $\psi := \max\{|\cdot|^2 - 4M^2, 0\}$ and let $\rho \in C^\infty(\mathbb{R}^2)$ be any smooth function such that $0 \leqslant \rho \leqslant 1$, $\operatorname{supp}(\rho) \subset B_{M/2}$ and

$$\int_{\mathbb{R}^2} \rho\, dx = 1.$$

Let $\eta := \psi * \rho$ and notice that

(1) η is convex and smooth;
(2) $\eta \equiv 0$ on $\overline{B_M}$;
(3) η has quadratic growth at infinity.

Define the strictly convex smooth functional $F := \sqrt{1+|\cdot|^2} + \eta : \mathbb{R}^2 \to \mathbb{R}$ and consider the smooth vector field $A = (A_1, A_2) := \nabla F : \mathbb{R}^2 \to \mathbb{R}^2$. It's easy to verify that there exist $\Lambda, \lambda > 0$ such that

(1) $|A_\beta(x)| \leqslant 2|x|$, for every $x \in \mathbb{R}^2$ and $\beta = 1, 2$;
(2) $|\partial_\alpha A_\beta(x)| \leqslant \Lambda$, for every $x \in \mathbb{R}^2$ and $\alpha, \beta = 1, 2$;
(3) $\displaystyle\sum_{\alpha,\beta=1}^{2} \partial_\alpha A_\beta(x)\xi_\alpha\xi_\beta \geqslant \lambda|\xi|^2$, for every $x \in \mathbb{R}^2$, $\xi \in \mathbb{R}^2$.

Step 1. We claim that $u \in H^2_{loc}(B_1)$ and $\partial_\gamma u \in H^1_{loc}(B_1)$ satisfies

$$\sum_{\beta=1}^{2} \int_{B_1} \nabla A_\beta(\nabla u) \cdot \nabla(\partial_\gamma u)\partial_\beta\varphi\, dx, \qquad \forall\, \varphi \in H^1_0(B_1), \tag{1.2}$$

for every $\gamma = 1, 2$. Indeed, by definition, $u \in H^1(B_1)$ satisfies

$$\int_{B_1} A(\nabla u) \cdot \nabla\varphi\, dx = 0, \qquad \forall\, \varphi \in H^1_0(B_1). \tag{1.3}$$

We want to proceed by using the difference quotients method. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ and fix any $\gamma = 1, 2$. Let $\varphi \in C^\infty_c(B_1)$ be any test function. For every $h > 0$ small enough, also $\varphi(\cdot - he_\gamma)$ belongs to $C^\infty_c(B_1)$. Thus, by testing equation (1.3) with $\varphi(\cdot - he_\gamma)$ and then changing the variable we get

$$0 = \int_{B_1} A(\nabla u) \cdot \nabla\varphi(\,\cdot - he_\gamma)\, dx$$

$$= \int_{B_1} \big(A(\nabla u(\,\cdot\,+he_\gamma)) - A(\nabla u)\big) \cdot \nabla\varphi \, dx. \tag{1.4}$$

For almost every $x \in B_1$, it holds that

$$\begin{aligned} A_\beta\big(\nabla u(x+h_\gamma)\big) - A_\beta\big(\nabla u(x)\big) &= \int_0^1 \frac{d}{dt}\bigg(A_\beta\big(t\nabla u(x+h_\gamma) + (1-t)\nabla u(x)\big)\bigg)\, dt \\ &= \int_0^1 \nabla A_\beta\big(t\nabla u(x+he_\gamma) + (1-t)\nabla u(x)\big) \cdot \nabla\big(u(x+he_\gamma) - u(x)\big)\, dt \\ &= \int_0^1 \nabla A_\beta\big(t\nabla u(x+he_\gamma) + (1-t)\nabla u(x)\big)\, dt \cdot \nabla\big(u(x+he_\gamma) - u(x)\big), \end{aligned}$$

for every $\beta = 1, 2$. Define

$$\tilde{A}^{\alpha\beta} := \int_0^1 \partial_\alpha A_\beta\big(t\nabla u(x+he_\gamma) + (1-t)\nabla u(x)\big)\, dt,$$

for every $\alpha, \beta = 1, 2$, and notice that

$$\begin{gathered} |\tilde{A}^{\alpha\beta}(x)| \leqslant \Lambda, \\ \tilde{A}^{\alpha\beta}(x)\xi_\alpha\xi_\beta \geqslant \lambda|\xi|^2, \end{gathered}$$

for every $\xi \in \mathbb{R}^2$. Moreover, (1.4) now reads

$$0 = \int_{B_1} \tilde{A}(x)\nabla\bigg(\frac{u(x+he_\gamma) - u(x)}{h}\bigg) \cdot \nabla\varphi(x)\, dx, \tag{1.5}$$

where $\tilde{A} := (\tilde{A}^{\alpha\beta})_{\alpha,\beta=1,2}$. Fix $x_0 \in B_1$ and let $r > 0$ be such that $B_r(x_0) \subset\subset B_1$. Let $\zeta \in C_c^\infty(\mathbb{R}^2)$ be such that $\operatorname{supp}(\zeta) \subset B_r(x_0)$, $0 \leqslant \zeta \leqslant 1$, $\zeta \equiv 1$ on $B_{r/2}(x_0)$ and $|\nabla\zeta| \leqslant 2/r$ on $\mathbb{R}^2$. By testing (1.5) with

$$\varphi := \zeta^2\bigg(\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg),$$

w and by exploiting the uniform ellipticity of the matrix $\tilde{A}$, we get

$$\begin{aligned} \int_{B_1} \zeta^2\bigg|\nabla\bigg(\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg)\bigg|^2 dx &\leqslant \frac{1}{\lambda}\int_{B_1} \zeta^2\tilde{A}^{\alpha\beta}\partial_\alpha\bigg(\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg)\partial_\beta\bigg(\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg)\, dx \\ &\leqslant \frac{2\Lambda}{\lambda}\int_{B_1} |\nabla\zeta|\bigg|\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg|\zeta\bigg|\nabla\bigg(\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg)\bigg|\, dx \\ &\leqslant \frac{2\Lambda}{\lambda}\bigg(\int_{B_1} |\nabla\zeta|^2\bigg|\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg|^2 dx\bigg)^{1/2} \cdot \\ &\quad \cdot \bigg(\int_{B_1} \zeta^2\bigg|\nabla\bigg(\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg)\bigg|^2 dx\bigg)^{1/2} \\ &\leqslant \frac{\lambda}{\Lambda}\int_{B_1} |\nabla\zeta|^2\bigg|\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg|^2 dx \\ &\quad + \frac{1}{2}\int_{B_1} \zeta^2\bigg|\nabla\bigg(\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg)\bigg|^2 dx. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \int_{B_1} \zeta^2\bigg|\nabla\bigg(\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg)\bigg|^2 dx &\leqslant \frac{2\lambda}{\Lambda}\int_{B_1} |\nabla\zeta|^2\bigg|\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg|^2 dx \\ &\leqslant \frac{8\lambda}{\Lambda r^2}\int_{B_r(x_0)} \bigg|\frac{u(\,\cdot\,+he_\gamma) - u}{h}\bigg|^2 dx \\ &\leqslant \frac{8\lambda C}{\Lambda r^2}\int_{B_1} |\nabla u|^2\, dx, \end{aligned} \tag{1.6}$$

where the last inequality follows since $u \in H^1(B_1)$. By Satz 8.3.1 - (iii) in Struwe's lecture notes, it follows that $\partial_\gamma u \in H^1_{loc}(B_1)$ and by arbitrariness of $\gamma = 1, 2$ we conclude $u \in H^2_{loc}(B_1)$. Moreover, estimate (1.6) allows to invoke dominated convergence in order to pass to the limit as $h \to +\infty$ in (1.5), which gives (1.2).

Step 2: Notice that (1.2) is the weak form of the uniformly elliptic equation in divergence form given by

$$\operatorname{div}\big(\nabla A(\nabla u)\nabla(\partial_\gamma u)\big) = 0. \tag{1.7}$$

By properties 2. and 3. of the smooth vector field A and since $u \in C^{1,\alpha}$ we get that (1.7) is a uniformly elliptic equation in divergence form with $C^{0,\alpha}$ coefficients. By Schauder theory, we get $\partial_\gamma u \in C^{1,\alpha}(B_1)$ for every $\gamma = 1, 2$. This gives $u \in C^{2,\alpha}_{loc}(B_1)$ and the statement follows. □

Notice that it was asbolutely necessary to further differentiate the equation (1.1) in order to use Schauder theory and infer further regularity. Indeed, equation (1.1) can be seen as a uniformly elliptic problem in divergence form with coefficients in $C^{0,\alpha}$. Such regularity for the coefficients is not enough in general to get $u \in C^{2,\alpha}$, as we can deduce by the following example.

Example 1.1. Let $a := 1 + |\cdot| : \mathbb{R} \to \mathbb{R}$ and consider the 1-dimensional elliptic problem in divergence form given by

$$(au')' = 0 \qquad \text{on } I := (-1, 1). \tag{1.8}$$

The coefficients of this problem as Lipschitz, thus $C^{0,\alpha}$ for every $\alpha \in (0, 1]$. Nevertheless, the function

$$u(x) := \begin{cases} \log(1 + x) & \text{on } [0, 1) \\ -\log(1 - x) & \text{on } (-1, 0) \end{cases}$$

is a weak solution of (1.8) on I that belongs to $C^{1,\alpha}(I)$ for every $\alpha \in (0, 1]$ and doesn't belong to $C^2_{loc}(I)$.

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