2.1. Weak derivative in $L^p(\Omega)$

(a) One implication is based on Hölder's inequality. For the converse implication, recall that $(L^q(\Omega))^*$ for $1 \leq q < \infty$ is isometrically isomorphic to $L^p(\Omega)$.

(b) Given $\varphi \in C_c^\infty(\mathbb{R})$ and $u = \chi_{]0,1[}$, compute $\int_{\mathbb{R}} u\,\varphi'\,dx$.

2.2. The ice-cream cone

(a) Fix $\varphi \in C_c^\infty(\Omega)$ and pick a small positive constant $0 < \varepsilon < 1$. Observe that

$$-\int_\Omega u(x,y)\frac{\partial\varphi}{\partial x}(x,y)\,dx\,dy = I_\varepsilon + J_\varepsilon,$$

with

$$I_\varepsilon := -\int_{\Omega\smallsetminus B_\varepsilon(0)} u(x,y)\frac{\partial\varphi}{\partial x}(x,y)\,dx\,dy = -\int_{\Omega\smallsetminus B_\varepsilon(0)} \Big(1-\sqrt{x^2+y^2}\Big)\frac{\partial\varphi}{\partial x}(x,y)\,dx\,dy$$

and

$$J_\varepsilon := -\int_{B_\varepsilon(0)} u(x,y)\frac{\partial\varphi}{\partial x}(x,y)\,dx\,dy = -\int_{B_\varepsilon(0)} \Big(1-\sqrt{x^2+y^2}\Big)\frac{\partial\varphi}{\partial x}(x,y)\,dx\,dy.$$

Integrate by parts in the integral I_ε. What happens at the limit $\varepsilon \to 0^+$?

2.3. Cantor function

(a) Measure the set $A_n = \{x \in]0,1[\mid u_n'(x) \neq 0 \text{ or } u_n'(x) \text{ does not exist}\}$.

(b) Construct φ_k such that u is constant in each component of the support of φ_k'.

(c) If the distributional derivative u' of u vanishes, then $u' = 0$ would be the weak first derivative of u in $L^1(]0,1[)$.

2.4. Symmetry of Green's function

If G is Green's function for Ω and $\varphi \in C^0(\Omega)$, then according to the Theorem

$$u(x) = \int_\Omega G(x,y)\varphi(y)\,dy \qquad \Rightarrow \begin{cases} -\Delta u = \varphi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

2.5. Green's function for the half-space

Given any fixed $x \in \mathbb{R}^n_+$, solve the boundary-value problem

$$\begin{cases} \Delta\phi^x = 0 & \text{in } \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y-x) & \text{for } y \in \partial\mathbb{R}^n_+ \end{cases}$$

explicitly by constructing ϕ^x with the help of Φ.

2.6. Green's function for an interval

(a) With Φ for $n = 1$, solve the boundary-value problem

$$\begin{cases} (\phi^x)'' = 0 & \text{in }]a,b[, \\ \phi^x(y) = \Phi(x-y) & \text{for } y \in \{a,b\}. \end{cases}$$

explicitly.

(b) Use the formula for $G(x,y)$ derived in part (a).