2.1. Weak derivative in $L^p(\Omega)$

(a) Let $u \in L^1_{\mathrm{loc}}(\Omega)$. Given $1 < p \leq \infty$, let $1 \leq q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $D^\alpha u$ exists as weak derivative in $L^p(\Omega)$. Let $\varphi \in C_c^\infty(\Omega)$ be arbitrary. Then,

$$\left|\int_\Omega u D^\alpha \varphi \, dx\right| = \left|(-1)^{|\alpha|} \int_\Omega (D^\alpha u)\varphi \, dx\right| \leq \|D^\alpha u\|_{L^p(\Omega)} \|\varphi\|_{L^q(\Omega)}$$

by Hölder's inequality which proves the first claim with constant $C = \|D^\alpha u\|_{L^p(\Omega)}$. Conversely, suppose

$$\forall \varphi \in C_c^\infty(\Omega): \quad \left|\int_\Omega u \, D^\alpha \varphi \, dx\right| \leq C\|\varphi\|_{L^q(\Omega)}.$$

Then, since $C_c^\infty(\Omega)$ is dense in $L^q(\Omega)$ for $q < \infty$, the map

$$f\colon \varphi \mapsto (-1)^{|\alpha|} \int_\Omega u \, D^\alpha \varphi \, dx$$

defines a continuous linear functional $f \in (L^q(\Omega))^*$. Since $(L^q(\Omega))^*$ for $1 \leq q < \infty$ is isometrically isomorphic to $L^p(\Omega)$, there exists $g \in L^p(\Omega)$ such that

$$\forall \varphi \in L^q(\Omega): \quad f(\varphi) = \int_\Omega g\varphi \, dx.$$

By definition of f it follows that $g \in L^p(\Omega)$ is the weak derivative $D^\alpha u$ of u.

(b) Let $u = \chi_{]0,1[}$ and $\varphi \in C_c^\infty(\mathbb{R})$. Then

$$\left|\int_\mathbb{R} u \, \varphi' \, dx\right| = \left|\int_0^1 \varphi' \, dx\right| = \left|\varphi(1) - \varphi(0)\right| \leq 2\|\varphi\|_{L^\infty(\mathbb{R})}.$$

The function u restricted to $\mathbb{R} \setminus \{0,1\}$ is differentiable with vanishing derivative. In particular, if u had a weak derivative $u' \in L^1_{\mathrm{loc}}(\mathbb{R})$, then $u' = 0$ almost everywhere. A contradiction arises for test functions $\varphi \in C_c^\infty(\mathbb{R})$ with $\varphi(0) \neq \varphi(1)$ via

$$0 = \int_\mathbb{R} u'\varphi \, dx = -\int_\mathbb{R} u \, \varphi' \, dx = -\int_0^1 \varphi' \, dx = \varphi(0) - \varphi(1).$$

2.2. The ice-cream cone

(a) Fix any $\varphi \in C_c^\infty(\Omega)$ and pick a small positive constant $0 < \varepsilon < 1$. Then, define

$$I_\varepsilon := -\int_{\Omega \smallsetminus B_\varepsilon(0)} u(x,y) \frac{\partial \varphi}{\partial x}(x,y) \, dx \, dy = -\int_{\Omega \smallsetminus B_\varepsilon(0)} \left(1 - \sqrt{x^2+y^2}\right) \frac{\partial \varphi}{\partial x}(x,y) \, dx \, dy$$

and

$$J_\varepsilon := -\int_{B_\varepsilon(0)} u(x,y)\frac{\partial \varphi}{\partial x}(x,y)\,dx\,dy = -\int_{B_\varepsilon(0)} \Big(1-\sqrt{x^2+y^2}\Big)\frac{\partial \varphi}{\partial x}(x,y)\,dx\,dy.$$

Clearly,

$$|J_\varepsilon| = \left|\int_{B_\varepsilon(0)} \Big(1-\sqrt{x^2+y^2}\Big)\frac{\partial \varphi}{\partial x}(x,y)\,dx\,dy\right| \le \pi\|\nabla\varphi\|_{L^\infty(\Omega)}\varepsilon^2 \to 0$$

as $\varepsilon \to 0^+$. On the other hand, since u is smooth on $\Omega \setminus B_\varepsilon(0)$, we can integrate by parts in the integral I_ε to get

$$\begin{aligned} I_\varepsilon &= \int_{\partial B_\varepsilon(0)} \Big(1-\sqrt{x^2+y^2}\Big)\varphi(x,y)\frac{x}{\varepsilon}\,d\sigma - \int_{\Omega\setminus B_\varepsilon(0)} \frac{x}{\sqrt{x^2+y^2}}\varphi(x,y)\,dx\,dy \\ &= (1-\varepsilon)\int_{\partial B_\varepsilon(0)} \varphi(x,y)\frac{x}{\varepsilon}\,d\sigma - \int_{\Omega\setminus B_\varepsilon(0)} \frac{x}{\sqrt{x^2+y^2}}\varphi(x,y)\,dx\,dy. \end{aligned}$$

Notice that

$$\left|(1-\varepsilon)\int_{\partial B_\varepsilon(0)} \varphi(x,y)\frac{x}{\varepsilon}\,d\sigma\right| \le 2\pi\|\varphi\|_{L^\infty(\Omega)}(1-\varepsilon)\varepsilon \to 0$$

as $\varepsilon \to 0^+$. Moreover, since

$$\begin{aligned} \left|\int_\Omega \frac{x}{\sqrt{x^2+y^2}}\varphi(x,y)\,dx\,dy\right| &\le \|\varphi\|_{L^\infty(\Omega)}\left(\int_0^{2\pi}|\cos\theta|\,d\theta\right)\cdot\left(\int_0^1 r dr\right) \\ &= 2\|\varphi\|_{L^\infty(\Omega)} < +\infty, \end{aligned}$$

by dominated convergence we get

$$-\int_{\Omega\setminus B_\varepsilon(0)} \frac{x}{\sqrt{x^2+y^2}}\varphi(x,y)\,dx\,dy \to -\int_\Omega \frac{x}{\sqrt{x^2+y^2}}\varphi(x,y)\,dx\,dy,$$

as $\varepsilon \to 0^+$. Thus,

$$I_\varepsilon + J_\varepsilon \to -\int_\Omega \frac{x}{\sqrt{x^2+y^2}}\varphi(x,y)\,dx\,dy.$$

But since

$$I_\varepsilon + J_\varepsilon = -\int_\Omega u(x,y)\frac{\partial\varphi}{\partial x}(x,y)\,dx\,dy,$$

for every $0<\varepsilon<1$, by uniqueness of the limit we obtain

$$-\int_\Omega u(x,y)\frac{\partial\varphi}{\partial x}(x,y)\,dx\,dy = -\int_\Omega \frac{x}{\sqrt{x^2+y^2}}\varphi(x,y)\,dx\,dy.$$

Since Ω has finite measure, it holds that $L^\infty(\Omega) \hookrightarrow L^p(\Omega)$ continuously for every $p \in [1,\infty)$, and so it follows that such weak partial derivative of u exists in $L^p(\Omega)$ for every $p \in [1,\infty]$ and is given by

$$\frac{\partial u}{\partial x}(x,y) = -\frac{x}{\sqrt{x^2+y^2}} \qquad \text{a.e. on } \Omega.$$

Analogous conclusions hold for the weak partial derivative with respect to y of u on Ω, which is given by

$$\frac{\partial u}{\partial y}(x,y) = -\frac{y}{\sqrt{x^2+y^2}} \qquad \text{a.e. on } \Omega.$$

(b) First, notice that

$$|\nabla u|^2 = \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 = 1 \qquad \text{a.e. on } \Omega.$$

Thus,

$$\|\nabla u\|_{L^p(\Omega)} = \pi^{1/p} \qquad \forall\, p \in [1,\infty)$$

and

$$\|\nabla u\|_{L^\infty(\Omega)} = 1.$$

2.3. Cantor function

(a) The set $A_n = \{x \in \,]0,1[\, : u_n'(x) \neq 0 \text{ or } u_n'(x) \text{ does not exist classically}\}$ is a union of relatively closed subintervals of equal length. With each iteration $n \rightsquigarrow n+1$ the number of intervals doubles but their length is divided by three. Therefore,

$$\lim_{n\to\infty} |A_n| = \lim_{n\to\infty} (\tfrac{2}{3})^n = 0.$$

By definition of u, we have $\{x \in \,]0,1[\, : u'(x) = 0 \text{ exists classically}\} \supset \,]0,1[\, \setminus A_n$ for every $n \in \mathbb{N}$. Thus, $u'(x) = 0$ in a set of full measure, i. e. for almost every $x \in \,]0,1[$.

(b) Given $2 \leq k \in \mathbb{N}$, let $\varphi_k \in C_c^\infty(]0,1[)$ be such that

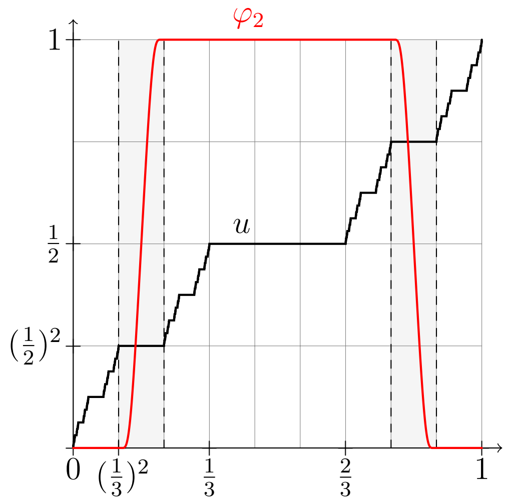
$$\varphi_k(x) = \begin{cases} 0 & \text{for } x \leq (\frac{1}{3})^k, \\ 1 & \text{for } 2(\frac{1}{3})^k \leq x \leq 1 - 2(\frac{1}{3})^k, \\ 0 & \text{for } x \geq 1 - (\frac{1}{3})^k. \end{cases}$$

Then, since

$$u(x)=\begin{cases}(\frac{1}{2})^k & \text{for } (\frac{1}{3})^k<x<2(\frac{1}{3})^k,\\ 1-(\frac{1}{2})^k & \text{for } 1-2(\frac{1}{3})^k<x<1-(\frac{1}{3})^k\end{cases}$$

and since $\varphi'(x)$ vanishes outside this range, there holds

$$\begin{aligned}-\int_0^1 u(x)\varphi_k'(x)\,dx &= -(\tfrac{1}{2})^k\int_{(\frac{1}{3})^k}^{2(\frac{1}{3})^k}\varphi'(x)\,dx-\Big(1-(\tfrac{1}{2})^k\Big)\int_{1-2(\frac{1}{3})^k}^{1-(\frac{1}{3})^k}\varphi'(x)\,dx\\ &= -(\tfrac{1}{2})^k+\Big(1-(\tfrac{1}{2})^k\Big)\xrightarrow{k\to\infty}1.\end{aligned}$$



Lastly, suppose the distributional derivative u' of u vanishes. Then $u'=0$ would be the weak first derivative of u in $L^1(]0,1[)$. However, $\|u'\|_{L^1(]0,1[)}=0$ is in contradiction to

$$\|u'\|_{L^1(]0,1[)}\geq\lim_{k\to\infty}\int_0^1 u'\varphi_k\,dx=-\lim_{k\to\infty}\int_0^1 u\varphi_k'\,dx=1.$$

2.4. Symmetry of Green's function

Let G be Green's function for $\Omega\subset\mathbb{R}^n$ and let $\varphi,\psi\in C_c^\infty(\Omega)$ be arbitrary. Consider the functions $u,v\colon\Omega\to\mathbb{R}$ given by

$$u(x)=\int_\Omega G(x,y)\varphi(y)\,dy,\qquad\qquad v(x)=\int_\Omega G(x,y)\psi(y)\,dy.$$

According to the Theorem about Green's function, they satisfy

$$\begin{cases}-\Delta u=\varphi & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}\qquad\qquad\begin{cases}-\Delta v=\psi & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega.\end{cases}$$

Therefore,

$$\begin{aligned}
&\int_\Omega \int_\Omega G(x,y)\varphi(y)\psi(x)\,dx\,dy - \int_\Omega \int_\Omega G(y,x)\varphi(y)\psi(x)\,dx\,dy \\
&= \int_\Omega u(x)\psi(x)\,dx - \int_\Omega v(y)\varphi(y)\,dy \\
&= -\int_\Omega u\Delta v\,dx + \int_\Omega v\Delta u\,dx = -\int_\Omega u\Delta v\,dx + \int_\Omega (\Delta v)u\,dx = 0,
\end{aligned}$$

where we used integration by parts and $v|_{\partial\Omega} = 0 = u|_{\partial\Omega}$ in the last line. Since φ and ψ are arbitrary, symmetry of G follows.

2.5. Green's function for the half-space

Given $x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n_+$, let $\overline{x} = (x_1, \ldots, x_{n-1}, -x_n)$ denote its reflection in the plane $\partial\mathbb{R}^n_+$. Let $\Phi\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be the fundamental solution of Laplace's equation as given on the problem set. Then the function

$$\phi^x(y) := \Phi(y - \overline{x}) = \Phi(y_1 - x_1, \ldots, y_{n-1} - x_{n-1}, y_n + x_n)$$

satisfies

$$\begin{cases} \Delta\phi^x = 0 & \text{in } \mathbb{R}^n_+, \\ \phi^x(y) = \Phi(y - x) & \text{for } y \in \partial\mathbb{R}^n_+ \end{cases}$$

because $y - \overline{x} \neq 0$ for every $y \in \mathbb{R}^n_+$ and since by symmetry of Φ

$$\forall y \in \partial\mathbb{R}^n_+ : \quad \Phi(y - x) = \Phi(\overline{y - x}) = \Phi(\overline{y} - \overline{x}) = \Phi(y - \overline{x}) = \phi^x(y).$$

Hence, Green's function for the upper half-space is

$$\begin{aligned}
G(x,y) &= \Phi(y - x) - \phi^x(y) = \Phi(y - x) - \Phi(y - \overline{x}) \\
&= \begin{cases} -\frac{1}{2\pi}\Big(\log|y - x| - \log|y - \overline{x}|\Big), & (n = 2) \\ \frac{1}{n(n-2)|B_1|}\Big(|y - x|^{2-n} - |y - \overline{x}|^{2-n}\Big), & (n \neq 2). \end{cases}
\end{aligned}$$

Remark. Since the domain $\mathbb{R}^n_+$ is unbounded, the representation formula (as given on the problem set) for solutions of the equation $-\Delta u = f$ in $\mathbb{R}^n_+$ with boundary data $u|_{\partial\mathbb{R}^n_+} = g$ has to be checked separately.

2.6. Green's function for an interval

(a) For $n = 1$, the fundamental solution of Laplace's equation is $\Phi\colon \mathbb{R}^1 \to \mathbb{R}$ given by

$$\Phi(x) = -\tfrac{1}{2}|x|.$$

Given $x \in \,]a,b[$, it remains to solve the boundary-value problem

$$\begin{cases} (\phi^x)'' = 0 & \text{in }]a,b[, \\ \phi^x(y) = -\frac{1}{2}|x-y| & \text{for } y \in \{a,b\}. \end{cases}$$

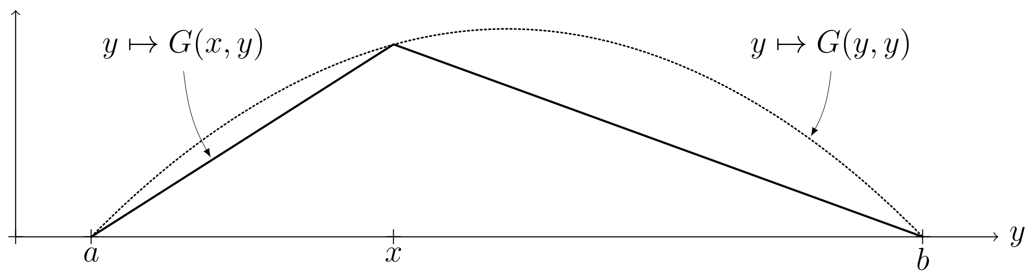
We obtain $\phi^x(y) = c_1 + c_2 y$ with constants $c_1, c_2 \in \mathbb{R}$ determined by the equations

$$\begin{aligned} -\tfrac{1}{2}(x-a) &= \phi^x(a) = c_1 + c_2 a & &\Rightarrow\; c_1 = -\tfrac{1}{2}(x-a) - c_2 a, \\ \tfrac{1}{2}(x-b) &= \phi^x(b) = c_1 + c_2 b & &\Rightarrow\; c_2(-a+b) = \tfrac{1}{2}(x-a) + \tfrac{1}{2}(x-b). \end{aligned}$$

Hence,

$$\begin{aligned} c_2 &= \frac{(x-a)+(x-b)}{2(b-a)}, \\ c_1 &= -\frac{x-a}{2} - \frac{(x-a)a + (x-b)a}{2(b-a)} = -\frac{(x-a)b + (x-b)a}{2(b-a)}, \end{aligned}$$

$$\begin{aligned} G(x,y) = \Phi(y-x) - c_1 - c_2 y &= -\frac{|y-x|}{2} + \frac{(x-a)(b-y) + (x-b)(a-y)}{2(b-a)} \\ &= \begin{cases} \frac{(x-b)(a-y)}{(b-a)} & \text{if } y \leq x, \\ \frac{(x-a)(b-y)}{(b-a)} & \text{if } y > x. \end{cases} \end{aligned}$$



(b) Let $f \in C^0([a,b])$ and $u(x) = \displaystyle\int_a^b G(x,y) f(y)\, dy$. Then,

$$\begin{aligned} u'(x) &= \int_a^b \frac{\partial G}{\partial x}(x,y) f(y)\, dy = \int_a^x \frac{(a-y)}{(b-a)} f(y)\, dy + \int_x^b \frac{(b-y)}{(b-a)} f(y)\, dy, \\ u''(x) &= \frac{(a-x)}{(b-a)} f(x) - \frac{(b-x)}{(b-a)} f(x) = \big((a-x) - (b-x)\big) \frac{f(x)}{(b-a)} = -f(x). \end{aligned}$$

Since $G(a,y) = 0 = G(b,y)$ for every $y \in \,]a,b[$, there holds $u(a) = 0 = u(b)$.