3.1. A closedness property

(a) Distinguish the cases $1 < p < \infty$ and $p = \infty$. In the first case apply the Eberlein–Šmulyan Theorem. In the second case, apply the Banach–Alaoglu Theorem.

(b) Is there a bounded sequence $(u_k)_{k\in\mathbb{N}}$ in $W^{1,1}(]-1,1[)$ such that $u_k \to \chi_{]0,1[}$ in L^1?

3.2. Fundamental solution of Laplace's equation in two dimensions

(a) Compute the partial derivatives classically using the chain rule.

(b) Argue that it suffices to compute $\int_{B_1(0)}|E|\,dx$ and $\int_{B_1(0)}|\nabla E|\,dx$.

(c) Integrate $E\Delta\varphi$ by parts over $\mathbb{R}^2 \setminus B_\varepsilon(0)$ and let $\varepsilon \to 0$.

(d) Recall $|x|^2 = z\overline{z}$.

(e) Use part (d).

3.3. Linear ODE with constant coefficients

(a) Apply the Riesz representation Theorem in the Hilbert space $\big(H^1_0(I), (\cdot,\cdot)_{H^1}\big)$.

(b) If $u \in H^1_0(I)$ is a weak solution, conclude that the function $u' \in L^2(I)$ has the weak derivative $(u - f) \in L^2(I)$.

(c) Solve for $v = u + v_0$, where u is the the solution from part (b) with a suitable right and side f and v_0 satisfies $v_0'' = 0$ and $v_0(a) = \alpha$, $v_0(b) = \beta$.

3.4. Linear ODE with variable coefficients

(a) To apply the Riesz representation Theorem, define a new scalar $\langle\cdot,\cdot\rangle$ product on $H^1_0(I)$ and prove that it is equivalent to the standard scalar product $(\cdot,\cdot)_{H^1_0}$

(b) If $u \in H^1_0(I)$ is a weak solution, conclude that the function $gu' \in L^2(I)$ has the weak derivative $(hu - f) \in L^2(I)$.

3.5. Extension operator of first and second order

Define $(Eu)(x)$ by odd reflection and candidates $g, h \in L^p_{\mathrm{loc}}(\mathbb{R})$ for the weak first and second derivatives of Eu. Then prove $(Eu)' = g$ and $(Eu)'' = h$ in the weak sense.

For the estimate, use $|u(0)| \le \|u\|_{L^\infty(\mathbb{R}_+)} \le C\|u\|_{W^{1,p}(\mathbb{R}_+)}$ which is Sobolev's inequality.

3.6. Extension operator of any order

(a) Find a linear system for $(a_1, \dots, a_k)$ involving a Vandermonde matrix.

(b) Proceed as in Problem 3.5. Notice that $\sum_{j=1}^k \left(-\frac{1}{j}\right)^\alpha a_j = 1$.