3.1. A closedness property

(a) Given $I :=]a,b[$ for $-\infty \leq a < b \leq \infty$ and $1 < p \leq \infty$, let $u \in L^p(I)$ and let $(u_k)_{k\in\mathbb{N}}$ be a bounded sequence in $W^{1,p}(I)$ satisfying $\|u_k - u\|_{L^p(I)} \to 0$ as $k \to \infty$. Let u_k' be the weak first derivative of u_k. By assumption, the sequence $(u_k')_{k\in\mathbb{N}}$ is bounded in $L^p(I)$.

Case $1 < p < \infty$. In this case, the space $L^p(I)$ is reflexive and the Eberlein–Šmulyan Theorem applies: $(u_k')_{k\in\mathbb{N}}$ has a subsequence which converges weakly in $L^p(I)$. Let $g \in L^p(I)$ be the corresponding weak limit and $\Lambda \subset \mathbb{N}$ the subsequence's indices. Since for any $\varphi \in C_c^\infty(I)$, the maps $L^p(I) \to \mathbb{R}$ given by $f \mapsto \int_I f\varphi\,dx$ or by $f \mapsto -\int_I f\varphi'\,dx$ are elements of $(L^p(I))^*$ and since $\|u_k - u\|_{L^p} \to 0$ implies $u_k \xrightarrow{\text{w}} u$, we have by definition of weak convergence

$$-\int_I u\varphi'\,dx = \lim_{\Lambda\ni k\to\infty}\Big(-\int_I u_k\varphi'\,dx\Big) = \lim_{\Lambda\ni k\to\infty}\Big(\int_I u_k'\varphi\,dx\Big) = \int_I g\varphi\,dx$$

for any $\varphi \in C_c^\infty(I)$. Hence, $g \in L^p(I)$ is indeed the weak derivative of $u \in L^p(I)$ and $u \in W^{1,p}(I)$ follows.

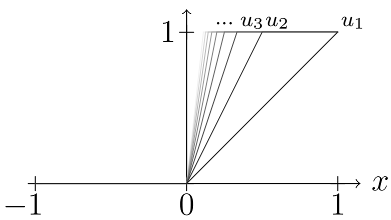
Case $p = \infty$. Since $L^1(I)$ is separable, the Banach–Alaoglu Theorem applies: $(u_k')_{k\in\mathbb{N}}$ being bounded in $L^\infty(I) \cong (L^1(I))^*$ has a subsequence (given by $\Lambda \subset \mathbb{N}$) which weak*-converges to some $g \in (L^1(I))^*$. For any $\varphi \in C_c^\infty(]0,1[) \subset L^1(]0,1[)$,

$$-\int_I u\varphi'\,dx = \lim_{\Lambda\ni k\to\infty}\Big(-\int_I u_k\varphi'\,dx\Big) = \lim_{\Lambda\ni k\to\infty}\Big(\int_I u_k'\varphi\,dx\Big) = \int_I g\varphi\,dx$$

follows as in part (a) with the only difference, that the last identity comes from weak*-convergence rather than weak convergence. Hence, $g \in (L^1(I))^* \cong L^\infty(I)$ is indeed the weak derivative of $u \in L^\infty(I)$ and $u \in W^{1,\infty}(I)$ follows.

(b) The assumption $p \neq 1$ in part (a) is necessary. Consider $I =]-1,1[$ and $u = \chi_{]0,1[} \in L^1(I)$. For every $k \in \mathbb{N}$ let $u_k\colon I \to \mathbb{R}$ be given by

$$u_k(x) = \begin{cases} 0, & \text{for } -1 < x \leq 0,\\ kx, & \text{for } 0 < x \leq \frac{1}{k},\\ 1, & \text{for } \frac{1}{k} < x \leq 1.\end{cases}$$



Then, $u_k \in W^{1,1}(I)$ with $\|u_k\|_{L^1} = 1 - \frac{1}{2k}$ and $\|u_k'\|_{L^1} = \frac{1}{k}k = 1$. Moreover, there holds $\|u_k - u\|_{L^1} = \frac{1}{2k} \to 0$ as $k \to \infty$. However, $u \notin W^{1,1}(I)$, otherwise u would have a continuous representative.

Remark. This is not a counterexample in the case $p > 1$, where $\|u_k'\|_{L^p} = (\frac{1}{k}k^p)^{\frac{1}{p}} \to \infty$.

3.2. Fundamental solution of Laplace's equation in two dimensions

(a) Given $j \in \{1,2\}$ and $x = (x_1, x_2) \in \mathbb{R} \setminus \{0\}$, we have

$$\begin{aligned} E(x) &= \frac{1}{2\pi} \log|x| = \frac{1}{4\pi} \log(x_1^2 + x_2^2), \\ \frac{\partial E}{\partial x_j}(x) &= \frac{2x_j}{4\pi|x|^2} = \frac{x_j}{2\pi|x|^2}. \end{aligned}$$

(b) Since E is represented smoothly away from the origin, it suffices to compute

$$\begin{aligned} \int_{B_1(0)} |E|\,dx &= 2\pi \int_0^1 (-\tfrac{1}{2\pi} \log r) r\,dr = -\int_0^1 r \log r\,dr = \tfrac{1}{4} r^2 (1 - 2\log r)\Big|_0^1 = \frac{1}{4}, \\ \int_{B_1(0)} |\nabla E|\,dx &= \int_{B_1(0)} \frac{|x|}{2\pi|x|^2}\,dx = 2\pi \int_0^1 \frac{r}{2\pi r^2}\,r\,dr = 1 \end{aligned}$$

in order to conclude $E \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $|\nabla E| \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$.

(c) Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ be arbitrary and $(r,\theta) \in \,]0,\infty[\, \times [0, 2\pi[$ polar coordinates in $\mathbb{R}^2$. Part (b) justifies the computation

$$\begin{aligned} \int_{\mathbb{R}^2} E \Delta\varphi\,dx &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon} E \Delta\varphi\,dx = \lim_{\varepsilon \to 0} \Big(-\int_{\partial B_\varepsilon} E \frac{\partial\varphi}{\partial r}\,d\sigma - \int_{\mathbb{R}^2 \setminus B_\varepsilon} \nabla E \cdot \nabla\varphi\,dx \Big) \\ &= -\int_{\mathbb{R}^2} \nabla E \cdot \nabla\varphi\,dx = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x}{|x|^2} \cdot \nabla\varphi\,dx \\ &= -\frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \frac{1}{r} \frac{\partial\varphi}{\partial r}\,r\,dr\,d\theta = \frac{1}{2\pi} \int_0^{2\pi} \varphi(0)\,d\theta = \varphi(0). \end{aligned}$$

(d) By definition,

$$\frac{\partial E}{\partial z} := \frac{1}{2} \Big(\frac{\partial E}{\partial x_1} - i \frac{\partial E}{\partial x_2} \Big) = \frac{x_1 - i x_2}{4\pi|x|^2} = \frac{\overline{z}}{4\pi z \overline{z}} = \frac{1}{4\pi z}.$$

(e) Let $f \in C^2(\mathbb{R}^2; \mathbb{C})$. Then, by symmetry of second derivatives,

$$4 \frac{\partial^2 f}{\partial z \partial \overline{z}} = \frac{\partial}{\partial x_1} \Big(\frac{\partial f}{\partial x_1} + i \frac{\partial f}{\partial x_2} \Big) - i \frac{\partial}{\partial x_2} \Big(\frac{\partial f}{\partial x_1} + i \frac{\partial f}{\partial x_2} \Big) = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} = \Delta f.$$

From part (d) we conclude

$$\frac{\partial}{\partial \overline{z}} \frac{1}{\pi z} = 4 \frac{\partial^2 E}{\partial \overline{z} \partial z} = \Delta E = \delta_0$$

in $\mathcal{D}'(\mathbb{R}^2)$.

3.3. Linear ODE with constant coefficients

(a) By definition, the space $H^1_0(I)$ is a closed subspace of the Hilbert space

$$\Big(H^1(I), (\cdot,\cdot)_{H^1}\Big), \qquad (u,v)_{H^1} := \int_I u'v'\,dx + \int_I uv\,dx.$$

In particular, $\Big(H^1_0(I), (\cdot,\cdot)_{H^1}\Big)$ is also Hilbertean. Given $f \in C^0(\overline{I})$, the map

$$\ell_f \colon H^1_0(I) \to \mathbb{R}, \qquad \ell_f(\varphi) := \int_I f(x)\varphi(x)\,dx$$

is a linear, continuous functional. In fact $|\ell_f(\varphi)| \leq \|f\|_{L^2}\|\varphi\|_{L^2} \leq \|f\|_{L^2}\|\varphi\|_{H^1}$. By the Riesz representation Theorem applied in the Hilbert space $(H^1_0(I), (\cdot,\cdot)_{H^1})$, there exists a unique $u \in H^1_0(I)$ satisfying

$$\forall \varphi \in H^1_0(I): \quad \int_I f\varphi\,dx =: \ell_f(\varphi) = (u,\varphi)_{H^1} = \int_I u'\varphi'\,dx + \int_I u\varphi\,dx. \tag{1}$$

(b) Let $u \in H^1_0(I)$ be the weak solution to the equation $-u'' + u = f$ in I found in part (a). By (1), we have in particular

$$\forall \varphi \in C^\infty_c(I): \quad -\int_I u'\varphi'\,dx = \int_I (u-f)\varphi\,dx.$$

Hence, the function $u' \in L^2(I)$ has the weak derivative $(u-f) \in L^2(I)$ and we conclude $u' \in H^1(I)$. Therefore, u' allows a continuous representative satisfying

$$u'(x) = u'(a) + \int_a^x (u-f)(t)\,dt. \tag{2}$$

Since $u \in H^1_0(I)$ allows a continuous representative and $f \in C^0(\overline{I})$, the right hand side of (2) is in $C^1(\overline{I})$. Finally, $u' \in C^1(\overline{I})$ implies $u \in C^2(\overline{I})$ as claimed.

(c) Let $g \in C^0(\overline{I})$. Let $\alpha, \beta \in \mathbb{R}$ and let $v_0 \in C^\infty(\overline{I})$ be given by

$$v_0(x) = \alpha + \frac{x-a}{b-a}(\beta - \alpha).$$

Let $f = g - v_0 \in C^0(\overline{I})$ and let $u \in H^1_0(I)$ be the solution of $-u'' + u = f$ found in part (a). By part (b), $u \in C^2(\overline{I})$. Moreover, $v := u + v_0 \in C^2(\overline{I})$ satisfies

$$\begin{cases} -v'' + v = -u'' - v_0'' + u + v_0 = -u'' + u + v_0 = f + v_0 = g, \\ \quad v(a) = u(a) + u_0(a) = u_0(a) = \alpha, \\ \quad v(b) = u(b) + u_0(b) = u_0(b) = \beta. \end{cases}$$

To prove uniqueness, let $\tilde{v} \in C^2(\overline{I})$ be another solution to the boundary-value problem

$$\begin{cases} -v'' + v = g & \text{in } I, \\ \quad v(a) = \alpha, & v(b) = \beta. \end{cases}$$

Then, the function $u := v - \tilde{v} \in C^2(\overline{I})$ satisfies $-u'' + u = 0$ with $u(a) = 0 = u(b)$. Moreover, since $u = u''$ integration by parts yields

$$\int_I u^2\,dx = \int_I u''u\,dx = -\int_I |u'|^2\,dx \leq 0$$

which implies $u = 0$ and hence $\tilde{v} = v$.

3.4. Linear ODE with variable coefficients

(a) Let $I = \,]a, b[$. Given $g \in C^1(\overline{I})$ and $h \in C^0(\overline{I})$ we assume that $g(x) \geq \lambda > 0$ and $h(x) \geq 0$ for every $x \in \overline{I}$ and define the new scalar product

$$\langle u, v\rangle := \int_I (g\,u'v' + h\,uv)\,dx$$

for all $u, v \in H^1_0(I)$. By assumption,

$$\langle u, u\rangle = \int_I (g\,|u'|^2 + h\,|u|^2)\,dx \geq \lambda \int_I |u'|^2\,dx$$

for any $u \in H^1_0(I)$. Moreover, using Poincaré's inequality,

$$\begin{aligned} \langle u, u\rangle &\leq \|g\|_{C^0} \int_I |u'|^2\,dx + \|h\|_{C^0} \int_I |u|^2\,dx \\ &\leq \Big(\|g\|_{C^0} + (b-a)^2\|h\|_{C^0}\Big) \int_I |u'|^2\,dx. \end{aligned}$$

Hence, $\langle\cdot,\cdot\rangle$ is equivalent to the standard scalar product $(u, v)_{H^1_0}$ on $H^1_0(I)$ given by

$$(u, v)_{H^1_0} := \int_I u'v'\,dx.$$

Hence, $(H^1_0(I), \langle\cdot,\cdot\rangle)$ is Hilbertean. Given $f \in C^0(\overline{I})$, the map

$$\ell_f\colon H^1_0(I) \to \mathbb{R}, \qquad\qquad \ell_f(\varphi) := \int_I f(x)\varphi(x)\,dx$$

is a linear, continuous functional. In fact, $|\ell_f(\varphi)| \leq \|f\|_{L^2}\|\varphi\|_{L^2} \leq (b-a)\|f\|_{L^2}\|\varphi\|_{H^1_0}$. By the Riesz representation Theorem applied in the Hilbert space $(H^1_0(I), \langle\cdot,\cdot\rangle)$, there exists a unique $u \in H^1_0(I)$ satisfying

$$\forall\varphi \in H^1_0(I): \quad \int_I f\varphi\,dx =: \ell_f(\varphi) = \langle u, \varphi\rangle = \int_I g\,u'\varphi' + h\,u\varphi\,dx \tag{3}$$

which is equivalent to being a weak solution of the equation

$$-(g\,u')' + h\,u = f \qquad \text{in } I. \tag{$\dagger$}$$

(b) Let $u \in H_0^1(I)$ be the weak solution to equation (†) found in (a). By (3), we have in particular,

$$\forall \varphi \in C_c^\infty(I): \quad -\int_I g u' \varphi' \, dx = \int_I (hu - f)\varphi \, dx.$$

Hence, the function gu' has the weak derivative $(hu - f) \in L^2(I)$ and we conclude $gu' \in H^1(I)$. Therefore, gu' allows a continuous representative satisfying

$$(gu')(x) = (gu')(a) + \int_a^x (hu - f)(t) \, dt. \tag{4}$$

Since $u \in H_0^1(I)$ allows a continuous representative and $h, f \in C^0(\overline{I})$, the right hand side of (4) is in $C^1(\overline{I})$. Finally, $gu' \in C^1(\overline{I})$ and $0 < \lambda \le g \in C^1(\overline{I})$ imply $u' \in C^1(\overline{I})$. Hence, $u \in C^2(\overline{I})$ as claimed.

3.5. Extension operator of first and second order

In order to extend $u \in W^{2,p}(\mathbb{R}_+)$ by "odd reflection", we define

$$(Eu)(x) := \begin{cases} u(x) & \text{for } x > 0, \\ 2u(0) - u(-x) & \text{for } x < 0, \end{cases} \qquad g(x) := \begin{cases} u'(x) & \text{for } x > 0, \\ u'(-x) & \text{for } x < 0, \end{cases}$$

$$h(x) := \begin{cases} u''(x) & \text{for } x > 0, \\ -u''(-x) & \text{for } x < 0, \end{cases}$$

where we extended the continuous representative of $u \in W^{2,p}(I)$ continuously at $x = 0$ to obtain the value $u(0)$.

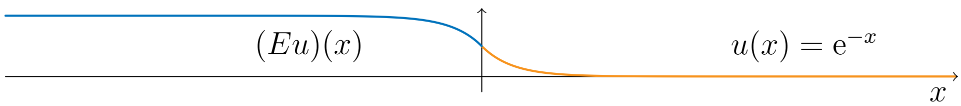


Figure 1: Extension by odd reflection.

Then, $(Eu), g, h \in L^p_{\mathrm{loc}}(\mathbb{R})$ because $u, u', u'' \in L^p(\mathbb{R}_+)$ and because the constant function $x \mapsto 2u(0)$ is in $L^p_{\mathrm{loc}}(\mathbb{R})$. We claim that g is the first and h the second weak derivative of Eu. Let $\varphi \in C_c^\infty(\mathbb{R})$ be arbitrary. Then,

$$\begin{aligned} -\int_{\mathbb{R}} (Eu)\varphi' \, dx &= -\int_{-\infty}^0 2u(0)\varphi'(x) - u(-x)\varphi'(x) \, dx - \int_0^\infty u(x)\varphi'(x) \, dx \\ &= -2u(0)\varphi(0) + \int_{-\infty}^0 u(-x)\varphi'(x) \, dx - \int_0^\infty u(x)\varphi'(x) \, dx \\ &= -\int_{-\infty}^0 -u'(-x)\varphi(x) \, dx + \int_0^\infty u'(x)\varphi(x) \, dx = \int_{\mathbb{R}} g\varphi \, dx, \end{aligned}$$

which proves that $g \in L^p_{\mathrm{loc}}(\mathbb{R})$ is the first weak derivative of Eu. Since $u' \in W^{1,p}(\mathbb{R}_+)$ we know from the lecture that h is the first weak derivative of g. Hence,

$$\int_{\mathbb{R}} (Eu)\varphi''\,dx = -\int_{\mathbb{R}} (Eu)'\varphi'\,dx = -\int_{\mathbb{R}} g\varphi'\,dx = \int_{\mathbb{R}} h\varphi\,dx$$

proves that $h \in L^p_{\mathrm{loc}}(\mathbb{R})$ is the weak second derivative of Eu and it follows that $E\colon W^{2,p}(\mathbb{R}_+) \to W^{2,p}_{\mathrm{loc}}(\mathbb{R})$ is well-defined. Let $K \subset \mathbb{R}$ be any compact subset. Then, since by Sobolev's embedding $|u(0)| \leq \|u\|_{L^\infty(\mathbb{R}_+)} \leq C\|u\|_{W^{1,p}(\mathbb{R}_+)}$, we may estimate

$$\|Eu\|_{L^p(K)} \leq 2|u(0)||K|^{\frac{1}{p}} + 2\|u\|_{L^p(\mathbb{R}_+)} \leq \Big(2C|K|^{\frac{1}{p}} + 2\Big)\|u\|_{W^{1,p}(\mathbb{R}_+)}.$$

With $\|(Eu)'\|_{L^p(K)} + \|(Eu)''\|_{L^p(K)} \leq 2\|u'\|_{L^p(\mathbb{R}_+)} + 2\|u''\|_{L^p(\mathbb{R}_+)} \leq 2\|u\|_{W^{2,p}(\mathbb{R}_+)}$, we obtain $\|Eu\|_{W^{2,p}(K)} \leq \tilde{C}\|u\|_{W^{2,p}(\mathbb{R}_+)}$ with constant $\tilde{C} = 2C|K|^{\frac{1}{p}} + 4$.

3.6. Extension operator of any order

(a) Let $k \in \mathbb{N}$. For $m \in \{0, \ldots, k-1\}$ and $p(x) = x^m$, we obtain the equation

$$\forall x \in \mathbb{R} \quad \sum_{j=1}^{k} a_j \Big(\frac{-x}{j}\Big)^m = x^m \qquad \Leftrightarrow \qquad \sum_{j=1}^{k} \frac{a_j}{j^m} = (-1)^m.$$

Equivalently,

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{k} \\ 1 & (\frac{1}{2})^2 & (\frac{1}{3})^2 & \cdots & (\frac{1}{k})^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & (\frac{1}{2})^{k-1} & (\frac{1}{3})^{k-1} & \cdots & (\frac{1}{k})^{k-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ \vdots \\ (-1)^{k-1} \end{pmatrix}.$$

The matrix A on the left hand side is a Vandermonde matrix. In particular,

$$\det A = \prod_{1 \leq i < j \leq k} \Big(\frac{1}{j} - \frac{1}{i}\Big) \neq 0$$

which implies that a unique solution $(a_1, \ldots, a_k) \in \mathbb{R}^k$ to the linear system exists. By linearity,

$$\sum_{j=1}^{k} a_j\, p\Big(\frac{-x}{j}\Big) = p(x).$$

holds not only for monomials $p(x) = x^m$ with $m \in \{0, \ldots, k-1\}$ but in fact for arbitrary polynomials of degree $k-1$.

(b) Let $k \in \mathbb{N}$ be fixed and $a_1, \dots, a_k$ as in part (a). Given $u \in W^{k,p}(\mathbb{R}_+)$, consider (Eu) as given on the problem set and

$$g_\alpha(x) := \begin{cases} D^\alpha u(x) & \text{for } x > 0, \\ \sum_{j=1}^k \left(-\frac{1}{j}\right)^\alpha a_j (D^\alpha u)\left(\frac{-x}{j}\right) & \text{for } x < 0 \end{cases}$$

for integers $0 \leq \alpha \leq k$. Then, $(Eu) \in L^p(\mathbb{R})$ since $u \in L^p(\mathbb{R}_+)$ and $g_\alpha \in L^p(\mathbb{R})$ since $(D^\alpha u) \in L^p(\mathbb{R}_+)$.

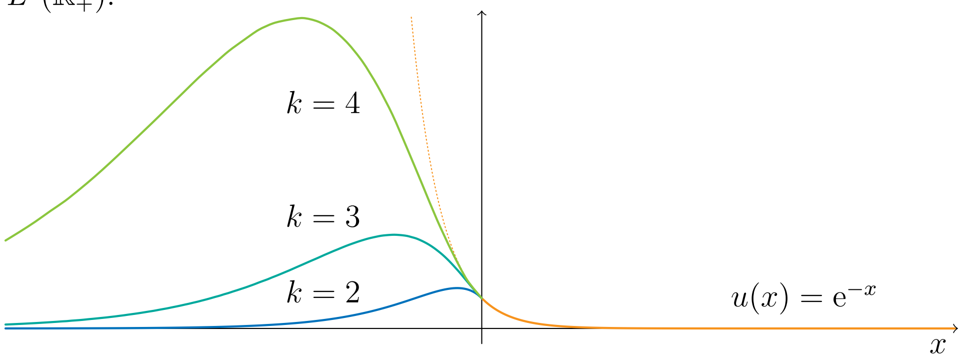


Figure 2: Extensions $(Eu)(x)$ of $u(x) = \mathrm{e}^{-x}$ for $k = 2, 3, 4$.

We prove by induction that g_α is the α-th weak derivative of (Eu). For $\alpha = 0$ we have $g_0 = Eu$ by construction. Suppose $D^\alpha(Eu) = g_\alpha$ for some $\alpha < k$. For $\varphi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}
&(-1)^{\alpha+1} \int_{\mathbb{R}} (Eu) D^{\alpha+1}\varphi \, dx = -\int_{\mathbb{R}} D^\alpha(Eu)\varphi' \, dx = -\int_{\mathbb{R}} g_\alpha \varphi' \, dx \\
&= -\sum_{j=1}^k \left(-\tfrac{1}{j}\right)^\alpha a_j \int_{-\infty}^0 (D^\alpha u)\left(-\tfrac{x}{j}\right)\varphi'(x) \, dx - \int_0^\infty (D^\alpha u)\varphi' \, dx \\
&= \sum_{j=1}^k \left(-\tfrac{1}{j}\right)^{\alpha+1} a_j \int_{-\infty}^0 (D^{\alpha+1}u)\left(-\tfrac{x}{j}\right)\varphi(x) \, dx - \sum_{j=1}^k \left(-\tfrac{1}{j}\right)^\alpha a_j (D^\alpha u)(0)\, \varphi(0) \\
&\quad + \int_0^\infty (D^{\alpha+1}u)\varphi \, dx + (D^\alpha u)(0)\, \varphi(0) \\
&= \int_{\mathbb{R}} g_{\alpha+1}\varphi \, dx + \left(1 - \sum_{j=1}^k \left(-\tfrac{1}{j}\right)^\alpha a_j\right)(D^\alpha u)(0)\, \varphi(0).
\end{aligned}$$

Since $\sum_{j=1}^k \left(-\frac{1}{j}\right)^\alpha a_j = 1$ was proven in part (a) (set $x = 1$ and $m = \alpha$), the claim $D^{\alpha+1}(Eu) = g_{\alpha+1}$ follows. Hence, $E \colon W^{k,p}(\mathbb{R}_+) \to W^{k,p}(\mathbb{R})$ is well-defined. Moreover, for any integer $0 \leq \alpha \leq k$,

$$\begin{aligned}
\|D^\alpha(Eu)\|_{L^p(\mathbb{R})} &\leq \|D^\alpha u\|_{L^p(\mathbb{R}_+)} + \Big\|\sum_{j=1}^k \left(-\tfrac{1}{j}\right)^\alpha a_j (D^\alpha u)\left(\tfrac{\cdot}{j}\right)\Big\|_{L^p(\mathbb{R}_+)} \\
&\leq \|D^\alpha u\|_{L^p(\mathbb{R}_+)} + \sum_{j=1}^k \frac{|a_j|}{j^\alpha} j^{\frac{1}{p}} \|D^\alpha u\|_{L^p(\mathbb{R}_+)} \leq C_{k,p}\|D^\alpha u\|_{L^p(\mathbb{R}_+)}.
\end{aligned}$$