Part I. Survival kit

6.1. Inextendible

Let $1 \leq p \leq \infty$. Consider the open set $\Omega =]-1,1[^2 \setminus ([0,1[\times \{0\})$.

(a) Prove that there does not exist an extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$.

(b) Where and why does the argument that you have seen in class fail in this situation?

6.2. Zero trace and H_0^1

Let $\Omega \subset \mathbb{R}^n$ be any open, bounded and non-empty subset of $\mathbb{R}^n$.

(a) Explain why any function $u \in H_0^1(\Omega)$ can be extended *by zero* to a function $\overline{u} \in H^1(\mathbb{R}^n)$.

Conversely, one wonders whether any function in $H^1(\mathbb{R}^n)$ which is zero outside Ω be restricted to a function in $H_0^1(\Omega)$. Through the following steps, we shall answer such a question (in the negative).

(b) Let $\Omega \subset \mathbb{R}^n$ be of class C^1. Let $v \in H^1(\mathbb{R}^n)$ satisfy $v(x) = 0$ for almost every $x \in \mathbb{R}^n \setminus \Omega$. Prove $v|_\Omega \in H_0^1(\Omega)$.

(c) Show that the assumption that Ω is of class C^1 cannot be dropped in part (b): find a bounded, connected, open set $\Omega \subset \mathbb{R}^2$ and $w \in H^1(\mathbb{R}^2)$ satisfying $w(x) = 0$ for almost every $x \in \mathbb{R}^2 \setminus \Omega$ such that $w|_\Omega \notin H_0^1(\Omega)$.

6.3. Ladyženskaja's inequality

As a warm-up to the following problem, please have a look at the *first proof* presented in the lecture notes for the Sobolev-Gagliardo-Nirenberg inequality (Satz 8.5.1 therein).

Adapt that argument to show that there is an embedding $H^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ and prove the estimate

$$\forall u \in H^1(\mathbb{R}^2): \quad \|u\|_{L^4(\mathbb{R}^2)}^4 \leq 4\|u\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2.$$

Part II. Project on compact embeddings

Recall that if $\Omega \subset \mathbb{R}^n$ is a *bounded* domain with boundary of class C^1 and $1 \leq p < n$, then Sobolev's embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is *compact* for any $1 \leq q < p^* := \frac{np}{n-p}$. Are the assumptions on the domain Ω necessary for compactness?

6.4. Non-compactness

Let $n \in \mathbb{N}$ and $1 \leq p \leq \infty$. Prove that the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ is *not* compact.

6.5. Compactness

(a) Let $n \in \mathbb{N}$ and $1 < p \leq n$. Let $\Omega \subset \mathbb{R}^n$ be possibly unbounded but of finite Lebesgue measure. Prove that then, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact.

(Recall that $W_0^{1,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ with respect to the $W^{1,p}(\Omega)$-norm.)

(b) Is the statement of part (a) still true, if the space $W_0^{1,p}(\Omega)$ is replaced by $W^{1,p}(\Omega)$?