6.1. Inextendible

(a) Let $\Omega = \,]-1,1[^2 \setminus ([0,1[\, \times \{0\})$ and let $u\colon \Omega \to \mathbb{R}$ be given by

$$u(x_1,x_2) := \begin{cases} x_1 & \text{if } x_1 > 0 \text{ and } x_2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

As shown in Problem 5.1, $u \in W^{1,\infty}(\Omega)$. Since Ω is bounded, $u \in W^{1,p}(\Omega)$ for any $1 \leq p \leq \infty$. Suppose, there exists an extension operator $E\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^2)$ such that $(Eu)|_\Omega = u$ almost everywhere in Ω. Let $Q := \,]-1,1[^2$ and $v := (Eu)|_Q$. Then $Eu \in W^{1,p}(\mathbb{R}^n)$ implies $v \in W^{1,p}(Q)$. Consequently, as shown in Problem 5.5, $(x_2 \mapsto v(x_1,x_2)) \in W^{1,p}(]-1,1[)$ for almost every $x_1 \in \,]-1,1[$. Moreover, since $[0,1[\,\times\{0\}$ has measure zero, $v(x_1,x_2) = u(x_1,x_2)$ for almost every $(x_1,x_2) \in Q$.

Hence, there exists some fixed $x_1 \in \,]\frac{1}{2},1[$ such that $(g\colon x_2 \mapsto v(x_1,x_2)) \in W^{1,p}(]-1,1[)$ and such that $g(x_2) = u(x_1,x_2)$ for almost every $x_2 \in \,]-1,1[$. By Sobolev's embedding in dimension one, g and hence $x_2 \mapsto u(x_1,x_2)$ has a representative in $C^0(]-1,1[)$. However, since we chose $x_1 > \frac{1}{2}$, this contradicts discontinuity of

$$x_2 \mapsto u(x_1,x_2) = \begin{cases} x_1 & \text{for } x_2 > 0, \\ 0 & \text{for } x_2 < 0. \end{cases}$$

(b) The issue is that Ω is not a topological manifold with boundary. In particular, every point $x \in [0,1] \times \{0\}$ belongs to the topological boundary of Ω but doesn't admit an open neighbourhood U such that $U \cap \Omega$ is even only homeomorphic to Q_+ (compare with the definition of open set with C^k boundary given in class).

6.2. Zero trace and H^1_0

(a) Let $u \in H^1_0(\Omega)$ and consider the extension by 0 of u on $\mathbb{R}^n$, which we denote by $\tilde{u}$. First notice that $\tilde{u} \in L^2(\mathbb{R}^n)$, since $||\tilde{u}||_{L^2(\mathbb{R}^n)} = ||u||_{L^2(\Omega)} < +\infty$.
Since $u \in H^1_0(\Omega)$, by definition there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty_c(\Omega)$ such that $u_k \to u$ in $H^1(\Omega)$. Fix any $\varphi \in C^\infty_c(\mathbb{R}^n)$. Notice that, by applying twice the dominated convergence theorem and since u_k vanishes on the boundary of Ω for every $k \in \mathbb{N}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \tilde{u}\nabla\varphi\,dx &= \int_\Omega u\nabla\varphi\,dx = \lim_{k\to+\infty}\int_\Omega u_k\nabla\varphi\,dx \\ &= -\lim_{k\to+\infty}\int_\Omega \nabla u_k\varphi\,dx = -\int_\Omega \nabla u\varphi\,dx = -\int_{\mathbb{R}^n}\Big(\nabla u\chi_\Omega\Big)\varphi\,dx. \end{aligned}$$

By arbitrariness of $\varphi \in C^\infty_c(\mathbb{R}^n)$, we get that $\tilde{u}$ admits weak gradient in $\mathbb{R}^n$, given by $\nabla\tilde{u} = \nabla u\chi_\Omega$. Since $||\nabla\tilde{u}||_{L^2(\mathbb{R}^n)} = ||\nabla u||_{L^2(\Omega)} < +\infty$, the statement follows.

(b) *Step 1.* The problem can be reduced to the following model case. Let

$$\begin{aligned}
Q &= \{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, |x'| < 1 \text{ and } |x_n| < 1\},\\
Q_+ &= \{x = (x', x_n) \in Q : \, x_n > 0\},\\
Q_0 &= \{x = (x', x_n) \in Q : \, x_n = 0\}.
\end{aligned}$$

Let $u \in H^1(Q)$ satisfy $u = 0$ in $Q \setminus Q_+$. Then we claim $\alpha u \in H^1_0(Q_+)$ for any $\alpha \in C^1_c(Q)$. Note that since α is compactly supported in Q, (αu) extends to a function in $H^1(\mathbb{R}^n)$ which allows mollification. Let $0 \le \rho \in C^\infty_c(B_1(0))$ satisfy

$$\operatorname{supp}(\rho) \subset \{(x', x_n) \in B_1(0) : \, \tfrac{1}{2} < x_n < 1\}, \qquad \int_{B_1(0)} \rho \, dx = 1$$

and let $\rho_m(x) := m^n \rho(mx)$ for $m \in \mathbb{N}$. Then, $\|\rho_m * (\alpha u) - (\alpha u)\|_{H^1} \to 0$ as $m \to \infty$. Moreover, if $x = (x', x_n) \in Q_+$ with $x_n < \frac{1}{4m}$ then $(\alpha u)(x - y) = 0$ whenever $y_n > \frac{1}{2m}$ because u vanishes outside Q_+. Hence, by choice of $\operatorname{supp}(\rho_m)$,

$$\Big(\rho_m * (\alpha u)\Big)(x) = \int_{\mathbb{R}^n} \rho_m(y) \, (\alpha u)(x - y) \, dy = 0 \quad \text{ if } x_n < \tfrac{1}{4m}$$

which implies $\rho_m * (\alpha u) \in C^\infty_c(Q_+)$ and therefore $\alpha u \in H^1_0(Q_+)$.

Step 2. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with boundary of class C^1. Since $\partial\Omega$ is compact and regular, there exist finitely many open sets $U_1, \ldots, U_N \subset \mathbb{R}^n$ and diffeomorphisms $h_k \colon Q \to U_k$ such that for every $k \in \{1, \ldots, N\}$

$$h_k(Q_+) = U_k \cap \Omega, \qquad h_k(Q_0) = U_k \cap \partial\Omega, \qquad \partial\Omega \subset \bigcup_{k=1}^N U_k.$$

Furthermore, there exists an open set $U_0 \subset \mathbb{R}^n$ such that $\overline{U_0} \subset \Omega$ and $\Omega \subset \bigcup_{k=0}^N U_k$. Let $(\varphi_k)_{k \in \{0, \ldots, N\}}$ be a corresponding partition of unity, i. e. a collection of smooth functions such that for every $k \in \{0, \ldots, N\}$

$$0 \le \varphi_k \le 1, \qquad \operatorname{supp}(\varphi_k) \subset U_k, \qquad \sum_{k=0}^N \varphi_k|_\Omega = 1.$$

Let $v \in H^1(\mathbb{R}^n)$ satisfy $v(x) = 0$ for almost every $x \in \mathbb{R}^n \setminus \Omega$. By Satz 8.3.3, $v \circ h_k \in H^1(Q)$ for $k \in \{1, \ldots, N\}$ and it satisfies $v \circ h_k = 0$ in $Q \setminus Q_+$. By Step 1, choosing $\alpha = \varphi_k \circ h_k$, we have $\varphi_k v \circ h_k \in H^1_0(Q_+)$ Let $w_k^{(m)} \in C^\infty_c(Q_+)$ be such that $\|w_k^{(m)} - \varphi_k v \circ h_k\|_{H^1(Q_+)} \to 0$ as $m \to \infty$. Moreover, since $\operatorname{supp}(\varphi_0) \subset U_0 \subset \Omega$, we can approximate $\varphi_0 v$ by $v_0^{(m)} \in C^\infty_c(\Omega)$ directly using mollification. Then, we have

$$w^{(m)} := v_0^{(m)} + \sum_{k=1}^N (w_k^{(m)} \circ h_k^{-1}) \in C^1_c(\Omega)$$

and since $v = \sum_{k=0}^{N} \varphi_k v$ in Ω by partition of unity,

$$\begin{aligned}\|w^{(m)} - v\|_{H^1(\Omega)} &\leq \|v_0^{(m)} - \varphi_0 v\|_{H^1(\Omega)} + \sum_{k=1}^{N} \Big\|w_k^{(m)} \circ h_k^{-1} - \varphi_k v\Big\|_{H^1(\Omega)} \\ &\leq \|v_0^{(m)} - \varphi_0 v\|_{H^1(\Omega)} + \sum_{k=1}^{N} C\Big\|w_k^{(m)} - \varphi_k v \circ h_k\Big\|_{H^1(Q_+)} \xrightarrow{m\to\infty} 0.\end{aligned}$$

Thus, we have approximated u with a sequence of functions in $C_c^1(\Omega)$. Since we know that functions in $C_c^1(\Omega)$ can be approximated in H^1 by functions in $C_c^\infty(\Omega)$, this concludes the proof of $v|_\Omega \in H_0^1(\Omega)$.

(c) Let $\Omega = \,]-1,1[^2 \setminus ([0,1[\, \times \{0\})$ and let $u \in C^\infty(\mathbb{R}^2)$ satisfy $u(x) = 1$ if $|x| < \frac{1}{2}$ and $u(x) = 0$ if $|x| > \frac{3}{4}$. Then $u \in H^1(\Omega)$ and $u(x) = 0$ for almost every $x \in \mathbb{R}^2 \setminus \Omega$. Towards a contradiction, suppose there exists a sequence of functions $u_m \in C_c^\infty(\Omega)$ such that $\|u_m - u\|_{H^1(\Omega)} \to 0$ as $m \to \infty$. Let $Q := \,]0,1[^2$ and $Q_0 = \,]0,1[\, \times \{0\}$. By Lemma 8.4.2 the trace operator $T\colon H^1(Q) \to L^2(Q_0)$ mapping $T\colon u \mapsto u|_{Q_0}$ is linear and continuous. In particular,

$$\|Tu_m - Tu\|_{L^2(Q_0)} \leq C\|u_m - u\|_{H^1(Q)} \xrightarrow{m\to\infty} 0.$$

Since $Q_0 \subset \partial\Omega$ implies $Tu_m = u_m|_{Q_0} = 0$, we obtain $u|_{Q_0} = 0$ in $L^2(Q_0)$. This however contradicts the fact that $u(x) = 1$ for $|x| < \frac{1}{2}$.

Consequently, the assumption that Ω is of class C^1 cannot be dropped in part (b).

6.3. Ladyženskaja's inequality

Sobolev's embedding (in the case $n = 2 = p$) states that the space $H^1(\mathbb{R}^2)$ embeds into $L^q(\mathbb{R}^2)$ for any $2 \leq q < \infty$, in particular for $q = 4$. The Sobolev inequality states

$$\exists C < \infty \quad \forall u \in H^1(\mathbb{R}^2): \quad \|u\|_{L^4(\mathbb{R}^2)} \leq C\|u\|_{H^1(\mathbb{R}^n)}.$$

In this special case, we claim that the following inequality also holds.

$$\forall u \in H^1(\mathbb{R}^2): \quad \|u\|_{L^4(\mathbb{R}^2)}^4 \leq 4\|u\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2.$$

Since $C_c^\infty(\mathbb{R}^2)$ is dense in $H^1(\mathbb{R}^2)$, it suffices to prove the inequality for $u \in C_c^\infty(\mathbb{R}^2)$. Let $u \in C_c^\infty(\mathbb{R}^2)$ and $(x_1, x_2) \in \mathbb{R}^2$. Then,

$$\begin{aligned}|u^2(x_1,x_2)| &= \left|\int_{-\infty}^{x_1} \frac{\partial u^2}{\partial x_1}(s,x_2)\,ds\right| = \left|\int_{-\infty}^{x_1} 2u(s,x_2)\frac{\partial u}{\partial x_1}(s,x_2)\,ds\right| \\ &\leq 2\int_{\mathbb{R}} |u(s,x_2)||\nabla u(s,x_2)|\,ds.\end{aligned}$$

Analogously,

$$|u^2(x_1,x_2)| \le 2\int_{\mathbb{R}} |u(x_1,t)||\nabla u(x_1,t)|\,dt.$$

Hence, by Fubini's theorem and the Cauchy–Schwarz inequality

$$\begin{aligned}
\|u\|_{L^4(\mathbb{R}^2)}^4 &= \int_{\mathbb{R}}\int_{\mathbb{R}} |u(x_1,x_2)|^4\,dx_1\,dx_2 = \int_{\mathbb{R}}\int_{\mathbb{R}} |u^2(x_1,x_2)||u^2(x_1,x_2)|\,dx_1\,dx_2 \\
&\le 2\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} |u(s,x_2)||\nabla u(s,x_2)| ds\Big)\int_{\mathbb{R}} |u^2(x_1,x_2)|\,dx_1\,dx_2 \\
&\le 4\int_{\mathbb{R}}\Big(\int_{\mathbb{R}} |u(s,x_2)||\nabla u(s,x_2)| ds\Big) dx_2 \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} |u(x_1,t)||\nabla u(x_1,t)| dt\Big) dx_1 \\
&= 4\Big(\int_{\mathbb{R}^2} |u||\nabla u|\,dx\Big)^2 \le 4\|u\|_{L^2(\mathbb{R}^2)}^2 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2.
\end{aligned}$$

6.4. Non-compactness

Let $n \in \mathbb{N}$ and $1 \le p \le \infty$. Let $u \in C_c^\infty(\mathbb{R}^n)$ satisfy $\|u\|_{W^{1,p}(\mathbb{R}^n)} = 1$. For any $k \in \mathbb{N}$, let $u_k(x) = u(x + ke_1)$, where $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$. Then $\|u_k\|_{W^{1,p}(\mathbb{R}^n)} = 1$ for every $k \in \mathbb{N}$. Towards a contradiction, suppose that the embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ is compact. Then the sequence $(u_k)_{k\in\mathbb{N}}$ allows a convergent subsequence in $L^p(\mathbb{R}^n)$, i. e. there exists an unbounded set $\Lambda_1 \subset \mathbb{N}$ and some $v \in L^p(\mathbb{R}^n)$ such that $\|u_k - v\|_{L^p} \to 0$ as $\Lambda_1 \ni k \to \infty$. Hence, there exists another subsequence denoted by $\Lambda_2 \subset \Lambda_1$ such that $u_k(x) \to v(x)$ converges pointwise as $\Lambda_2 \ni k \to \infty$ for almost every $x \in \mathbb{R}^n$. However, since the support of u is a bounded subset of $\mathbb{R}^n$, we have pointwise convergence $u_k(x) \to 0$ as $k \to \infty$ for every $x \in \mathbb{R}^n$. Therefore, $v = 0$ almost everywhere. A contradiction arises from

$$0 < \|u\|_{L^p(\mathbb{R}^n)} = \|u_k\|_{L^p(\mathbb{R}^n)} \xrightarrow{\Lambda_1 \ni k \to \infty} \|v\|_{L^p(\mathbb{R}^n)} = 0.$$

6.5. Compactness

(a) Let $n \in \mathbb{N}$ and $1 < p \leq n$. Let $\Omega \subset \mathbb{R}^n$ be of finite Lebesgue measure. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,p}(\Omega)$ satisfying $\|u_k\|_{W^{1,p}(\Omega)} \leq C_1$ for every $k \in \mathbb{N}$. In particular, $u_k \in W_0^{1,p}(\Omega)$ can be extended *by zero* to a function $\overline{u}_k \in W^{1,p}(\mathbb{R}^n)$. Thus, $\|\overline{u}_k\|_{W^{1,p}(\mathbb{R}^n)} \leq C_1$ for every $k \in \mathbb{N}$. Since $1 < p < \infty$, the space $W^{1,p}(\mathbb{R}^n)$ is reflexive and there exists a subsequence $(\overline{u}_k)_{k\in\Lambda_1\subset\mathbb{N}}$ converging weakly to some $v \in W^{1,p}(\mathbb{R}^n)$.

For any $R > 0$, the embedding $W^{1,p}(B_R) \hookrightarrow L^p(B_R)$ is compact. Hence, a subsequence $(\overline{u}_k|_{B_R})_{k\in\Lambda_R\subset\Lambda_1}$ converges in $L^p(B_R)$. Restricting to nested subsequences for each $R \in \mathbb{N}$ and choosing a diagonal sequence, we find $\Lambda_2 \subset \Lambda_1$ (independently of R) such that $(\overline{u}_k|_{B_R})_{k\in\Lambda_2}$ converges in $L^p(B_R)$ for any $R \in \mathbb{N}$. Moreover, the limit must coincide with $v|_{B_R}$ by uniqueness of weak limits: both, weak convergence in $W^{1,p}$ and norm-convergence in L^p imply weak convergence in L^p.

We claim that $\|\overline{u}_k - v\|_{L^p(B_R)} \to 0$ as $\Lambda_2 \ni k \to \infty$ implies that $\|u_k - v\|_{L^p(\Omega)} \to 0$.

If $p < n$, then Sobolev's embedding $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ implies

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B_R} |\overline{u}_k|^p\,dx &= \int_{\Omega\setminus B_R} |u_k|^p\,dx \\
&\leq \Big(\int_{\Omega\setminus B_R} |\overline{u}_k|^{p^*}\,dx\Big)^{\frac{p}{p^*}} \Big(\int_{\Omega\setminus B_R} 1^{\frac{n}{p}}\,dx\Big)^{\frac{p}{n}} && \text{(Hölder's inequality)} \\
&\leq \Big(\int_{\mathbb{R}^n} |\overline{u}_k|^{p^*}\,dx\Big)^{\frac{p}{p^*}} |\Omega\setminus B_R|^{\frac{p}{n}} \\
&\leq C_{n,p}\,\|\nabla\overline{u}_k\|^p_{L^p(\mathbb{R}^n)} |\Omega\setminus B_R|^{\frac{p}{n}} && \text{(Sobolev's inequality, } p<n) \\
&\leq C_{n,p}\,C_1 |\Omega\setminus B_R|^{\frac{p}{n}}.
\end{aligned}$$

If $p = n$, then $W^{1,n}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ for any $n \leq q < \infty$, in particular for $q = 2n$. Thus,

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B_R} |\overline{u}_k|^n\,dx &= \int_{\Omega\setminus B_R} |u_k|^n\,dx \\
&\leq \Big(\int_{\Omega\setminus B_R} |\overline{u}_k|^{2n}\,dx\Big)^{\frac{1}{2}} \Big(\int_{\Omega\setminus B_R} 1^2\,dx\Big)^{\frac{1}{2}} && \text{(Hölder's inequality)} \\
&\leq \Big(\int_{\mathbb{R}^n} |\overline{u}_k|^{2n}\,dx\Big)^{\frac{1}{2}} |\Omega\setminus B_R|^{\frac{1}{2}} \\
&\leq C_{n,p}\,\|\overline{u}_k\|^n_{W^{1,n}(\mathbb{R}^n)} |\Omega\setminus B_R|^{\frac{1}{2}} && \text{(Sobolev's inequality, } p=n) \\
&\leq C_{n,p}\,C_1 |\Omega\setminus B_R|^{\frac{1}{2}}.
\end{aligned}$$

The same estimates also hold for $v \in W^{1,p}(\mathbb{R}^n)$ in place of $\overline{u}_k$. Let $\varepsilon > 0$ be arbitrary. Since $|\Omega| < \infty$, the estimates above imply that there exists some $R_\varepsilon \in \mathbb{N}$ such that

$$\forall k \in \mathbb{N}: \quad \|\overline{u}_k\|^p_{L^p(\mathbb{R}^n\setminus B_{R_\varepsilon})} < \varepsilon, \qquad\qquad \|v\|^p_{L^p(\mathbb{R}^n\setminus B_{R_\varepsilon})} < \varepsilon.$$

Moreover, as shown above, there exists $N_\varepsilon \in \mathbb{N}$ such that $\|\overline{u}_k - v\|^p_{L^p(B_{R_\varepsilon})} < \varepsilon$ for every $\Lambda_2 \ni k \geq N_\varepsilon$. The claim follows from

$$\begin{aligned}\|u_k - v\|^p_{L^p(\Omega)} &\leq \|\overline{u}_k - v\|^p_{L^p(\mathbb{R}^n)} = \|\overline{u}_k - v\|^p_{L^p(\mathbb{R}^n\setminus B_{R_\varepsilon})} + \|\overline{u}_k - v\|^p_{L^p(B_{R_\varepsilon})} \\ &\leq \Big(\|\overline{u}_k\|_{L^p(\mathbb{R}^n\setminus B_{R_\varepsilon})} + \|v\|_{L^p(\mathbb{R}^n\setminus B_{R_\varepsilon})}\Big)^p + \|\overline{u}_k - v\|^p_{L^p(B_{R_\varepsilon})} \\ &< (2^p+1)\varepsilon.\end{aligned}$$

Hence, the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega)$ is indeed compact.

(b) The embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is *not* always compact if $\Omega \subset \mathbb{R}^n$ is of finite measure but unbounded. An example for $n \geq 2$ is the domain $\Omega \subset \mathbb{R}^n$ given by

$$\Omega := \bigcup_{m=2}^{\infty} B_{\frac{1}{m}}(me_1), \qquad |\Omega| = |B_1| \sum_{m=2}^{\infty} m^{-n} < \infty,$$

where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Let $u_k = k^{\frac{n}{p}} \chi_{B_{\frac{1}{k}}(ke_1)}$. This function is constant on the k-th connected component of Ω and zero on the rest of Ω. Hence, $u_k \in W^{1,p}(\Omega)$ with

$$\|u_k\|^p_{W^{1,p}(\Omega)} = \|u_k\|^p_{L^p(\Omega)} = |B_{\frac{1}{k}}|k^n = |B_1| \qquad \forall k \geq 2.$$

Suppose, there exists a subsequence $(u_k)_{k\in\Lambda_1\subset\mathbb{N}}$ converging in $L^p(\Omega)$ to some $v \in L^p(\Omega)$. Then there exists a subsequence $(u_k)_{k\in\Lambda_2\subset\Lambda_1}$ such that $u_k(x) \to v(x)$ pointwise as $\Lambda_2 \ni k \to \infty$ for almost every $x \in \Omega$. By construction however, $u_k(x) \to 0$ as $k \to \infty$ for every $x \in \Omega$. Hence, $v = 0$ almost everywhere. A contradiction arises from

$$0 < \|u_k\|_{L^p(\Omega)} \xrightarrow{\Lambda_1 \ni k \to \infty} \|v\|_{L^p(\Omega)} = 0.$$