7.1. Completeness of Campanato spaces

Let $(u_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in the Campanato space $(\mathcal{L}^{p,\lambda}(\Omega), \|\cdot\|_{\mathcal{L}^{p,\lambda}(\Omega)})$. In particular, $(u_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in the space $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ which is complete. Hence there exists $v \in L^p(\Omega)$ such that

$$\lim_{k\to\infty} \|u_k - v\|_{L^p(\Omega)} = 0.$$

It remains to prove $v \in \mathcal{L}^{p,\lambda}(\Omega)$ and $\lim\limits_{k\to\infty}[u_k - v]_{\mathcal{L}^{p,\lambda}} = 0$. Let $x_0 \in \Omega$ and $0 < r < r_0$. Since by Hölder's inequality

$$|(u_m)_{x_0,r} - v_{x_0,r}|^p = \left| ⨍_{\Omega\cap B_r(x_0)} u_m - v\,dx \right|^p \leq ⨍_{\Omega\cap B_r(x_0)} |u_m - v|^p\,dx \xrightarrow{m\to\infty} 0,$$

we conclude that $(u_m - (u_m)_{x_0,r})$ converges to $(v - v_{x_0,r})$ in $L^p(\Omega \cap B_r(x_0))$ as $m \to \infty$. In particular,

$$\begin{aligned} r^{-\frac{\lambda}{p}}\|v - v_{x_0,r}\|_{L^p(\Omega\cap B_r(x_0))} &= \lim_{m\to\infty} r^{-\frac{\lambda}{p}}\|u_m - (u_m)_{x_0,r}\|_{L^p(\Omega\cap B_r(x_0))} \\ &\leq \limsup_{m\to\infty}[u_m]_{\mathcal{L}^{p,\lambda}}. \end{aligned} \tag{1}$$

Since $(u_m)_{m\in\mathbb{N}}$ being Cauchy in $\mathcal{L}^{p,\lambda}(\Omega)$ implies that (1) is finite, and since $x_0 \in \Omega$ and $0 < r < r_0$ are arbitrary, $[v]_{\mathcal{L}^{p,\lambda}} < \infty$ follows. Hence, $v \in \mathcal{L}^{p,\lambda}(\Omega)$.

Let $\varepsilon > 0$. By assumption, there exists $N_\varepsilon \in \mathbb{N}$ such that $[u_n - u_m]_{\mathcal{L}^{p,\lambda}} < \varepsilon$ for all $n, m \geq N_\varepsilon$ which implies that for every $x_0 \in \Omega$ and for all $0 < r < r_0$ and $n, m \geq N_\varepsilon$

$$r^{-\frac{\lambda}{p}}\|u_n - (u_n)_{x_0,r} - u_m + (u_m)_{x_0,r}\|_{L^p(\Omega\cap B_r(x_0))} < \varepsilon. \tag{2}$$

As in (1), we may pass to the limit $m \to \infty$ in (2) and obtain

$$r^{-\frac{\lambda}{p}}\|u_n - (u_n)_{x_0,r} - v + v_{x_0,r}\|_{L^p(\Omega\cap B_r(x_0))} < \varepsilon \tag{3}$$

for every $n \geq N_\varepsilon$. Since $x_0 \in \Omega$ and $0 < r < r_0$ are arbitrary, we conclude $[u_n - v]_{\mathcal{L}^{p,\lambda}} < \varepsilon$ for every $n \geq N_\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\|u_n - v\|_{\mathcal{L}^{p,\lambda}} \to 0$ as $n \to \infty$ follows.

7.2. Vanishing weak gradient

Let $1 \leq p \leq \infty$. Let $u \in W^{1,p}(\Omega)$ satisfy $\nabla u = 0 \in L^p(\Omega)$. Since $\Omega \subset \mathbb{R}^n$ is connected and bounded of class C^1, there exists $\varepsilon_0 > 0$ such that $\Omega_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ is connected for every $0 < \varepsilon < \varepsilon_0$. Fix $0 < \varepsilon_1 < \varepsilon_0$ and let $\rho_\varepsilon \in C_c^\infty(B_\varepsilon(0))$ be a standard mollifier for $0 < \varepsilon < \varepsilon_1$. Then the mollification $u_\varepsilon = u * \rho_\varepsilon$ is well-defined and smooth in Ω_{ε_1} and satisfies $\nabla u_\varepsilon = 0$ in Ω_{ε_1} classically, i. e. u_ε is constant in Ω_{ε_1}.

Moreover, $\|u-u_\varepsilon\|_{L^1(\Omega_{\varepsilon_1})}\to 0$ as $\varepsilon\to 0$ implies that the constants $u_\varepsilon|_{\Omega_{\varepsilon_1}}$ converge:

$$\Big|u_\varepsilon - \fint_{\Omega_{\varepsilon_1}} u\,dx\Big| = \Big|\fint_{\Omega_{\varepsilon_1}} u_\varepsilon - u\,dx\Big| \leq \fint_{\Omega_{\varepsilon_1}} |u_\varepsilon - u|\,dx \xrightarrow{\varepsilon\to 0} 0.$$

Since $\|u-u_\varepsilon\|_{L^1(\Omega_{\varepsilon_1})}\to 0$ implies pointwise convergence almost everywhere on a subsequence, we obtain $u(x_0)=\fint_{\Omega_{\varepsilon_1}} u\,dx$ by uniqueness of limits for almost every $x_0\in\Omega_{\varepsilon_1}$. Letting $\varepsilon_1\to 0$ completes the proof.

Remark. The statement generalises to arbitrary connected, open sets $\Omega\subset\mathbb{R}^n$.

7.3. Hölder continuity of functions in $W^{2,n}$

Let $0<\alpha<1$ be arbitrary. Let $u\in W^{2,n}(\mathbb{R}^n)$. Then u and $\partial_j u$ are in $W^{1,n}(\mathbb{R}^n)$ for any $j\in\{1,\dots,n\}$. For any $n\leq p<\infty$, especially for $p=\frac{n}{1-\alpha}$, we have the embedding $W^{1,n}(\mathbb{R}^n)\hookrightarrow L^p(\mathbb{R}^n)$. Hence, u and $\partial_j u$ are in $L^{\frac{n}{1-\alpha}}(\mathbb{R}^n)$ for any $j\in\{1,\dots,n\}$, which shows $u\in W^{1,\frac{n}{1-\alpha}}(\mathbb{R}^n)$. We conclude via the embedding $W^{1,\frac{n}{1-\alpha}}(\mathbb{R}^n)\hookrightarrow C^{0,\alpha}(\mathbb{R}^n)$.

7.4. Uniform bounds on functions in $W^{n,1}$

Let $u\in C_c^\infty(\mathbb{R}^n)$ and let $x=(x_1,\dots,x_n)\in\mathbb{R}^n$ be arbitrary. Then,

$$\begin{aligned}
u(x_1,\dots,x_n) &= \int_{-\infty}^{x_1}\frac{\partial u}{\partial x_1}(s_1,x_2,\dots,x_n)\,ds_1\\
&= \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\frac{\partial^2 u}{\partial x_2\partial x_1}(s_1,s_2,x_2,\dots,x_n)\,ds_2\,ds_1\\
&= \int_{-\infty}^{x_1}\dots\int_{-\infty}^{x_n}\frac{\partial^n u}{\partial x_n\dots\partial x_1}(s_1,\dots,s_n)\,ds_n\dots ds_1,\\
\Rightarrow\ |u(x)| &\leq \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty}\Big|\frac{\partial^n u}{\partial x_n\dots\partial x_1}(s_1,\dots,s_n)\Big|\,ds_n\dots ds_1 \leq \|u\|_{W^{n,1}(\mathbb{R}^n)}.
\end{aligned}$$

Since $x\in\mathbb{R}^n$ is arbitrary,

$$\forall u\in C_c^\infty(\mathbb{R}^n):\quad \|u\|_{L^\infty(\mathbb{R}^n)}\leq\|u\|_{W^{n,1}(\mathbb{R}^n)} \tag{4}$$

follows. The inequality (4) remains true for arbitrary $u\in W^{n,1}(\mathbb{R}^n)$ by density of $C_c^\infty(\mathbb{R}^n)$ in $W^{n,1}(\mathbb{R}^n)$. Indeed, given $u\in W^{n,1}(\mathbb{R}^n)$, let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $C_c^\infty(\mathbb{R}^n)$ such that $\|u_k-u\|_{W^{n,1}(\mathbb{R}^n)}\to 0$ as $k\to\infty$. Since inequality (4) implies $\|u_k-u_m\|_{L^\infty(\mathbb{R}^n)}\leq\|u_k-u_m\|_{W^{n,1}(\mathbb{R}^n)}$ the sequence $(u_k)_{k\in\mathbb{N}}$ is Cauchy in $L^\infty(\mathbb{R}^n)$ and hence convergent to some v in $L^\infty(\mathbb{R}^n)$. In particular, $u_k(x)\to v(x)$ converges pointwise for almost every $x\in\mathbb{R}^n$ as $k\to\infty$. Moreover, since $\|u_k-u\|_{L^n(\mathbb{R}^n)}\to 0$ implies pointwise convergence almost everywhere on a subsequence, $v=u$ almost everywhere follows by uniqueness of limits. Passing to the limit $k \to \infty$ in $\|u_k\|_{L^\infty(\mathbb{R}^n)} \leq \|u_k\|_{W^{n,1}(\mathbb{R}^n)}$ proves the claim.

7.5. A variant of the Poincaré inequality

Let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded of class C^1. Let μ be the Lebesgue measure on Ω. Let $1 \leq p < \infty$ and $\alpha > 0$. Towards a contradiction, we assume that there exists a sequence $(u_k)_{k\in\mathbb{N}}$ in $W^{1,p}(\Omega)$ such that for every $k \in \mathbb{N}$

$$\mu(\{x \in \Omega :\ u_k(x) = 0\}) \geq \alpha, \qquad \|u_k\|_{L^p(\Omega)} > k\|\nabla u_k\|_{L^p(\Omega)}. \tag{5}$$

Without loss of generality, we may assume $\|u_k\|_{L^p(\Omega)} = 1$ for every $k \in \mathbb{N}$. Otherwise we replace u_k with $\|u_k\|^{-1}_{L^p(\Omega)} u_k$ which preserves both inequalities (5). As a consequence, $\|u_k\|_{W^{1,p}(\Omega)} < 1 + \frac{1}{k}$ for any $k \in \mathbb{N}$ which shows that $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$.

The Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact for any $1 \leq p < \infty$. In the case $1 \leq p < n$, compactness holds because $p < p^*$. In the case $n \leq p < \infty$ compactness holds because $W^{1,p}(\Omega) \hookrightarrow W^{1,n}(\Omega) \hookrightarrow L^p(\Omega)$, where the second embedding is compact (Korollar 8.5.1) and the first embedding continuous by Hölder's inequality.

Hence, there exists a subsequence $(u_k)_{k\in\Lambda\subset\mathbb{N}}$ and some $v \in L^p(\Omega)$ such that

$$\lim_{\Lambda \ni k \to \infty} \|u_k - v\|_{L^p(\Omega)} = 0.$$

Moreover, convergence in L^p implies that there exists a subsequence $(u_k)_{k\in\Lambda'\subset\Lambda}$ such that $u_k(x) \to v(x)$ converges pointwise for almost every $x \in \Omega$ as $\Lambda' \ni k \to \infty$. Since $\|\nabla u_k\|_{L^p(\Omega)} \to 0$ as $k \to \infty$ by (5) and since the space $W^{1,p}(\Omega)$ is complete, we have $v \in W^{1,p}(\Omega)$ satisfying $\nabla v = 0$ which according to problem 7.2 implies that v has a constant representative. (Here it is crucial that Ω is connected.) If we prove

$$\mu(\{x \in \Omega :\ v(x) = 0\}) \geq \alpha > 0,$$

then $v \equiv 0$ would follow which would contradict $\forall k \in \mathbb{N} : \|u_k\|_{L^p(\Omega)} = 1$. Let

$$A_m := \bigcup_{k\in\Lambda',\, k\geq m} \{x \in \Omega :\ u_k(x) = 0\},$$

$$A := \bigcap_{m=1}^{\infty} A_m.$$

Then, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ and since $\mu(A_1) \leq \mu(\Omega) < \infty$ and $\mu(A_m) \geq \alpha$ we have

$$\mu(A) = \lim_{m\to\infty} \mu(A_m) \geq \alpha.$$

Since we have pointwise convergence $u_k(x) \to v(x)$ as $\Lambda' \ni k \to \infty$ for almost every $x \in A$ and since by construction, $u_k(x) = 0$ for infinitely many $k \in \Lambda'$ and every $x \in A$, we conclude $v(x) = 0$ for almost every $x \in A$. Therefore,

$$\mu(\{x \in \Omega :\ v(x) = 0\}) \geq \mu(A) \geq \alpha.$$

7.6. Explosion of the Poincaré constant

For $k \in \mathbb{N}$ let $\Omega_k = Q_+ \cup A_k \cup Q_-$ and $u\colon \Omega_k \to \mathbb{R}$ be given by

$$\begin{aligned} Q_+ &= \,]1,3[\, \times \,]{-1},1[, \\ A_k &= [-1,1] \times \,]{-\tfrac{1}{k}},\tfrac{1}{k}[, \\ Q_- &= \,]{-3},-1[\, \times \,]{-1},1[, \end{aligned} \qquad u(x_1,x_2) = \begin{cases} 1, & \text{if } (x_1,x_2) \in Q_+, \\ x_1, & \text{if } (x_1,x_2) \in A_k, \\ -1, & \text{if } (x_1,x_2) \in Q_-. \end{cases}$$

Since u is Lipschitz continuous, $u \in W^{1,\infty}(\Omega)$ and because Ω is bounded $u \in W^{1,p}(\Omega)$ for any $1 \le p < \infty$. Moreover, $u_{\Omega_k} = \fint_{\Omega_k} u\,dx = 0$ and

$$\int_{\Omega_k} |u - u_{\Omega_k}|^p\,dx = \int_{\Omega_k} |u|^p\,dx \ge 8, \qquad \int_{\Omega_k} |\nabla u|^p\,dx = \int_{A_k} 1\,dx = \frac{4}{k}.$$

Combining these two facts with the assumed Poincaré inequality, we have $8 \le C(\Omega_k)\frac{4}{k}$. Therefore, $C(\Omega_k) \ge 2k \to \infty$ as $k \to \infty$.