Part I. Survival kit

9.1. Elliptic equations in non-divergence form

Definition. Let $\Omega \subset \mathbb{R}^n$ be open. Let $a_{ij} \colon \Omega \to \mathbb{R}$ be measurable functions for every $i, j \in \{1, \dots, n\}$. A differential operator L in non-divergence form

$$Lu = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

is called (uniformly) elliptic in Ω, if there exists $\lambda > 0$ such that for almost every $x \in \Omega$ and every $\xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi^i \xi^j \geq \lambda |\xi|^2. \tag{1}$$

(a) Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Let $f \in L^2(\Omega)$. Let $a_{ij} \in C^2(\overline{\Omega})$ and $c \in C^0(\overline{\Omega})$ satisfy

$$\forall x \in \Omega : \quad c(x) \geq \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 a_{ij}}{\partial x_j \partial x_i}(x).$$

Let a_{ij} also satisfy (1). Prove that then, there exists a unique $u \in H^1_0(\Omega)$ such that

$$\forall \varphi \in H^1_0(\Omega) : \quad \sum_{i,j=1}^{n} \int_\Omega a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \frac{\partial a_{ij}}{\partial x_i} \frac{\partial u}{\partial x_j} \varphi \, dx + \int_\Omega c u \varphi \, dx = \int_\Omega f \varphi \, dx. \tag{2}$$

In this case, u is called weak solution of $-Lu + cu = f$.

(b) Under the same assumptions as in part (a) prove that a classical solution $u \in C^2(\Omega) \cap H^1_0(\Omega)$ of $-Lu + cu = f$ satisfies (2).

We state the following interior regularity estimate without proof.

Theorem (Interior regularity)**.** *Given $k \in \mathbb{N}$ let $f \in H^k(\Omega)$ and $c \in C^{k+1}(\Omega)$. Let $a_{ij} \in C^{k+2}(\Omega)$ satisfy (1). Suppose, $u \in H^1_0(\Omega)$ is a weak solution of*

$$-\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + cu = f$$

as defined in part (a). Then, $u \in H^{k+2}_{loc}(\Omega)$ and for each $\Omega' \subset\subset \Omega$ there holds

$$\|u\|_{H^{k+2}(\Omega')} \leq C\Big(\|f\|_{H^k(\Omega)} + \|u\|_{L^2(\Omega)}\Big)$$

with a finite constant C depending only on k, Ω, Ω' and the coefficients a_{ij} and c.

In particular, if $k > \frac{n}{2}$ the weak solution u is a classical solution.

9.2. The reflection Lemma towards boundary regularity

Let $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$ be the upper half-space and let $f \in L^2(\mathbb{R}^n_+)$. Let $u \in H^1_0(\mathbb{R}^n_+)$ with $\operatorname{supp}(u) \subset\subset \mathbb{R}^n$ be a weak solution to

$$-\Delta u = f \quad \text{ in } \mathbb{R}^n_+.$$

Using the notation $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, let $\overline{u}, \overline{f} \colon \mathbb{R}^n \to \mathbb{R}$ be given by

$$\overline{u}(x) = \begin{cases} u(x', x_n) & \text{if } x_n > 0, \\ -u(x', -x_n) & \text{if } x_n < 0, \end{cases}$$

$$\overline{f}(x) = \begin{cases} f(x', x_n) & \text{if } x_n > 0, \\ -f(x', -x_n) & \text{if } x_n < 0. \end{cases}$$

Show that $\overline{u} \in H^1(\mathbb{R}^n)$ with compact support and prove that $\overline{u}$ is a weak solution of

$$-\Delta \overline{u} = \overline{f} \quad \text{ in } \mathbb{R}^n.$$

9.3. Horizontal derivatives

Given $u \in H^2(\mathbb{R}^n_+) \cap H^1_0(\mathbb{R}^n_+)$ prove that $\dfrac{\partial u}{\partial x_i} \in H^1_0(\mathbb{R}^n_+),\ \forall i \in \{1, \ldots, n-1\}$.

Part II. Project on the bilaplacian

9.4. Properties of the bilaplacian

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary and

$$\Xi := \{u \in H^4(\Omega) \cap H_0^1(\Omega) : \Delta u \in H_0^1(\Omega)\}.$$

(a) Prove that the following operator is bijective.

$$\begin{aligned} \Delta^2 \colon \Xi &\to L^2(\Omega) \\ u &\mapsto \Delta(\Delta u) \end{aligned}$$

(b) Given $f \in L^2(\Omega)$, let $u \in \Xi$ satisfy $\Delta^2 u = f$. Prove that

$$\forall \varphi \in \Xi : \quad \int_\Omega u \Delta^2 \varphi \, dx = \int_\Omega f \varphi \, dx. \tag{3}$$

(c) Assume that $u, f \in L^2(\Omega)$ satisfy (3). Prove that $u \in \Xi$.

9.5. Weak solutions to the bilaplace equation

Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.

(a) Prove that

$$\langle u, v \rangle := \int_\Omega \Delta u \Delta v \, dx$$

defines a scalar product on $H^2(\Omega) \cap H_0^1(\Omega)$ which is equivalent to the standard scalar product $(\cdot, \cdot)_{H^2(\Omega)}$.

(b) Show that $\big(H^2(\Omega) \cap H_0^1(\Omega), \langle \cdot, \cdot \rangle\big)$ is a Hilbert space.

(c) Prove that given $f \in L^2(\Omega)$ there is a unique $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying

$$\forall v \in H^2(\Omega) \cap H_0^1(\Omega) : \quad \int_\Omega \Delta u \Delta v \, dx = \int_\Omega f v \, dx.$$

Show that in fact $u \in \Xi := \{u \in H^4(\Omega) \cap H_0^1(\Omega) : \Delta u \in H_0^1(\Omega)\}$ and $\Delta^2 u = f$.