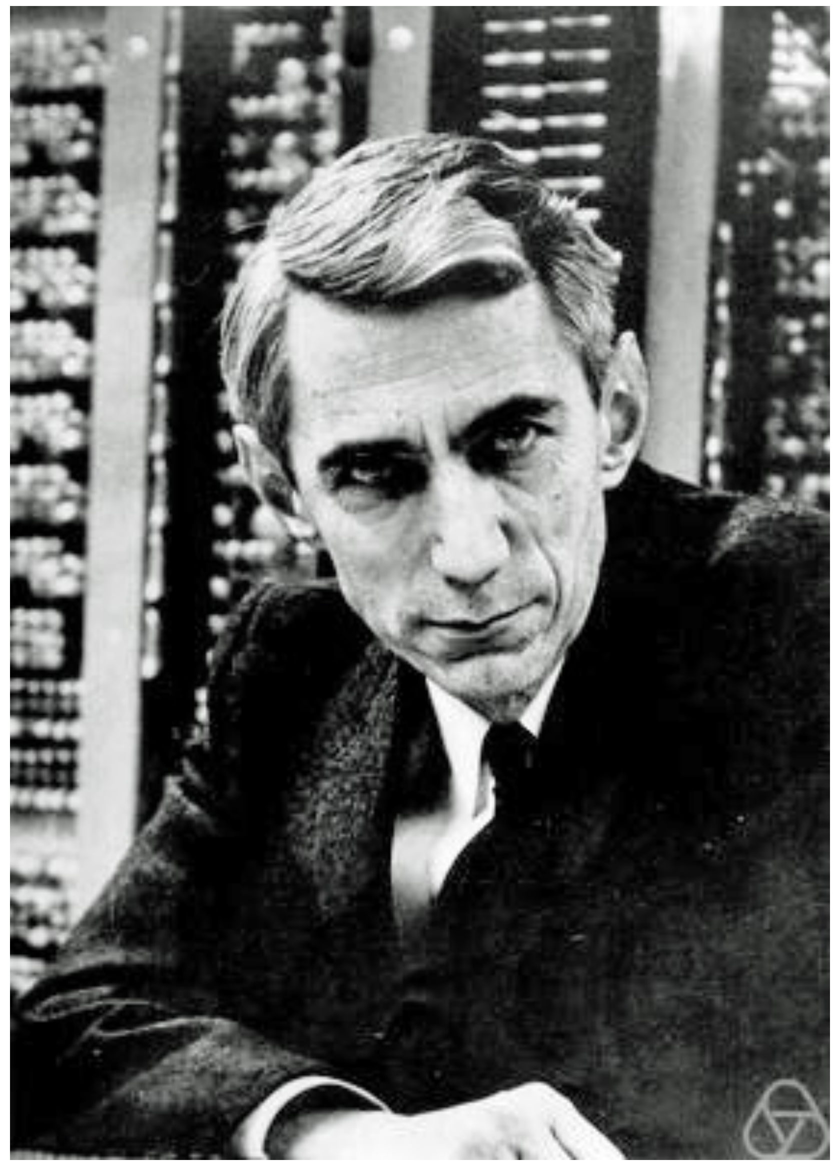
Shannon Capacity of Graphs



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Introductory example

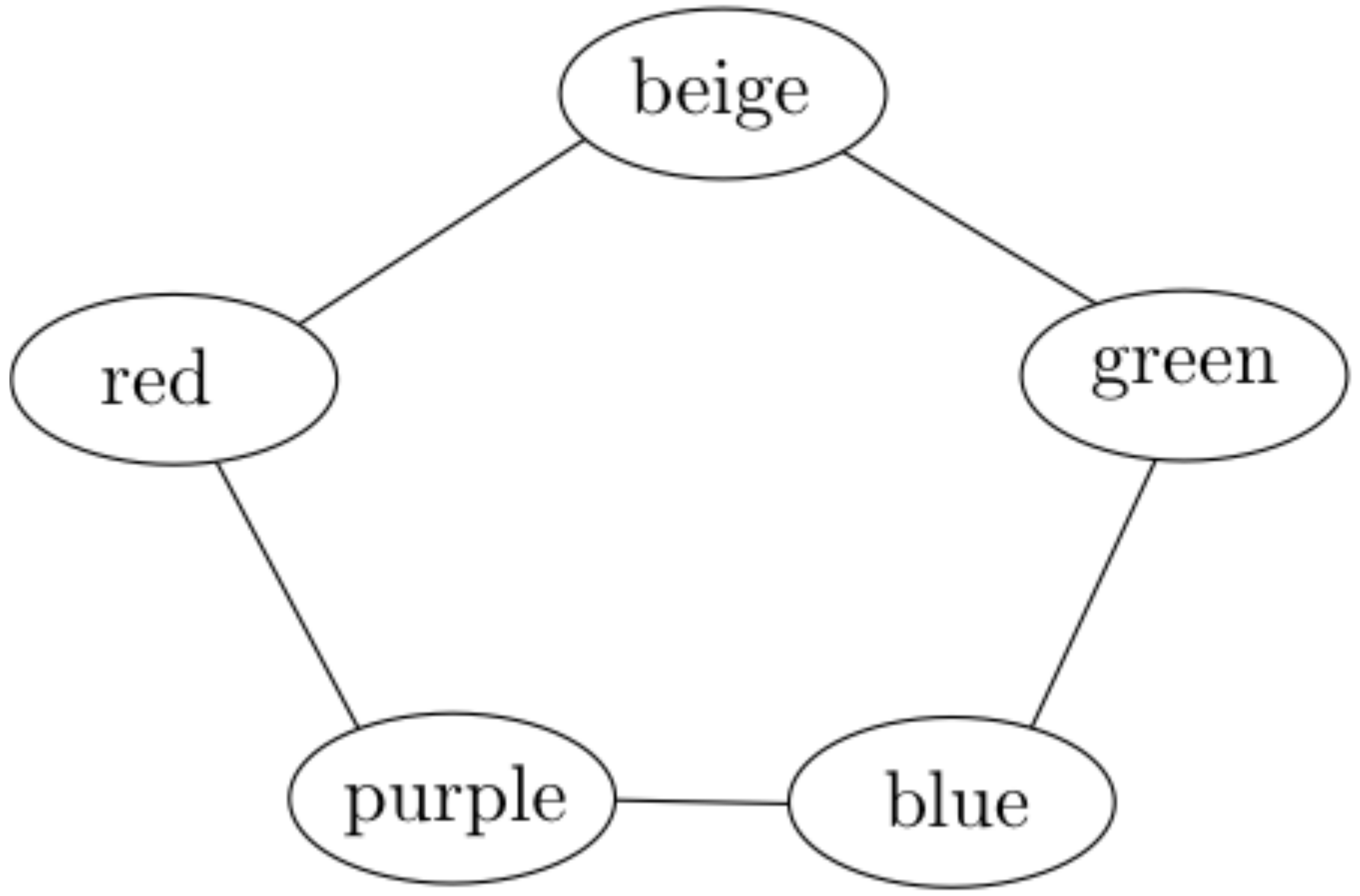
Given a Graph $G = (V, E)$ where the two nodes are connected iff they can be confused when sent through a noisy channel.

General Question: How many messages can we send on average per day with guarantee that no two messages will be confused?

In one day we can send $\alpha(G)$ many messages (where $\alpha(G)$ denotes the size of the largest independent set).

So in k days we can send at least $\alpha(G)^k$ messages.

Can we do better?



	the first day	the second day
message 1	red	red
message 2	beige	green
message 3	green	purple
message 4	blue	beige
message 5	purple	blue

Definiton of Shannon Capacity

Then Shannon Capacity of a graph G is defined as: $\Theta(G) = \sup\{\alpha_k(G)^{\frac{1}{k}} : k = 1,2,...\}$ where $\alpha_k(G)$ denotes the maximum size of a set of messages of length k with no interchangeable pair ($\alpha_1(G) = \alpha(G)$).

So for a sufficiently large k we can approximately send $\Theta(G)^k$ messages per day.

The strong product $H \cdot H'$ of two graphs H, H' is defined as $V(H \cdot H') = V(H) \times V(H')$
$E(H \cdot H') = \{\{(u, u'), (v, v')\} : (u = v$ or $\{u, v\} \in E(H)$ and at the same time $u' = v'$ or $\{u', v'\} \in E(H)\}$

Let G^k denote the strong product of k copies of G then $\alpha_k(G) = \alpha(G^k)$ and therefore $\Theta(G) = \sup\{\alpha(G^k)^{\frac{1}{k}} : k = 1,2,...\}$

For any G^k we have $\alpha(G^k)^{\frac{1}{k}} \leq \Theta(G)$

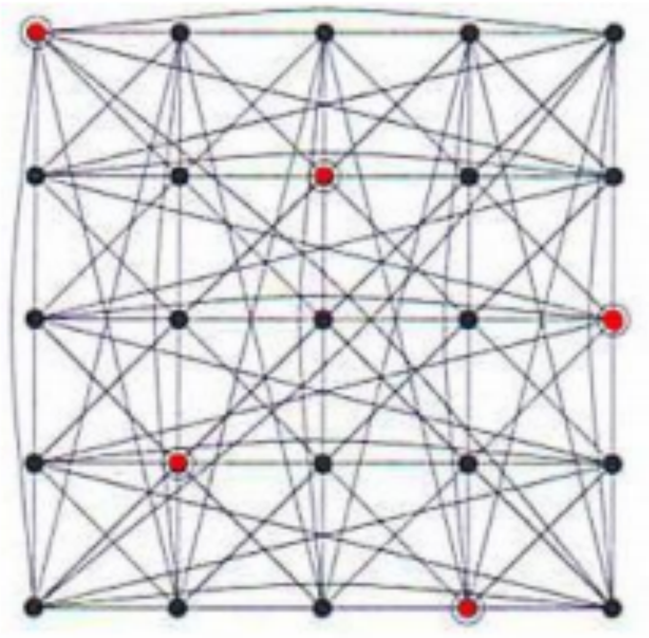


Figure 1.1: Strong product $C_5 \boxtimes C_5$

First observation by Shannon

A very useful trick is the so called adjacency reducing mapping $\gamma : V \to A \subseteq V$ which has the property that if i, j are non adjacent then $\gamma(i), \gamma(j)$ must also be non adjacent.

If we have a zero error code then we may apply such a map to get another zero error code.

If a adjacency reducing mapping can be found to a disjoint union of nodes then the Shannon capacity of the graph equals to the number of nodes in that disjoint union.

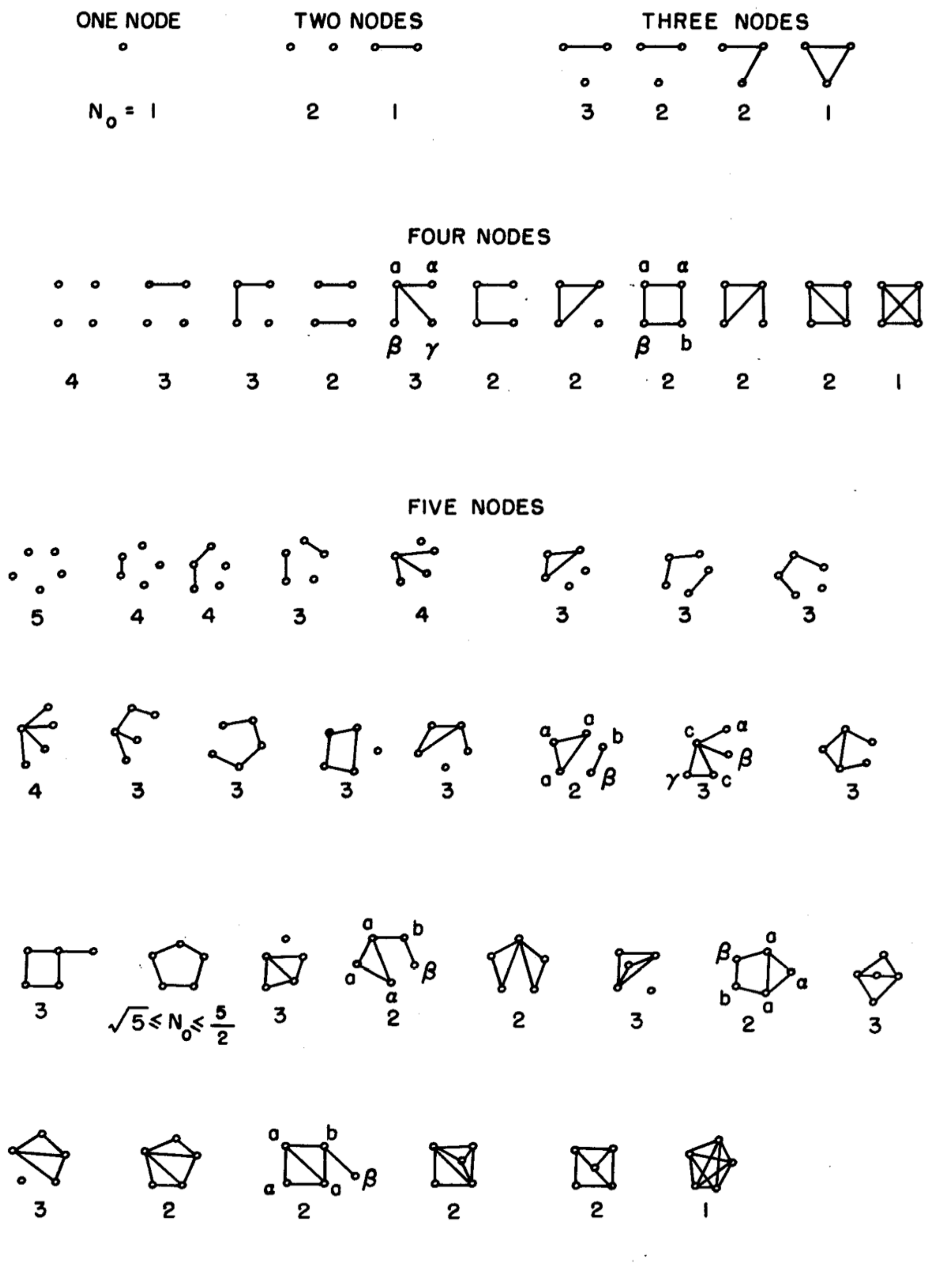


Fig. 4 - All graphs with 1, 2, 3, 4, 5 nodes and the corresponding N_0 for channels with these as adjacency graphs (note $C_0 = \log N_0$)

Lovaz theta function

Let $H = (V, E)$ be any graph then ab orthogonal representation of H is a mapping $\rho : V \to \mathbb{R}^n$ for some n that assigns a unit vector i.e $||\rho(v)|| = 1$ to every vertex such that the following holds: If two distinct vertices u, v are not connected by an edge, then the corresponding vectors are orthogonal $\langle \rho(u), \rho(v) \rangle = 0$

For any graph H and orthogonal representation ρ we define

$\vartheta(H, \rho) := max_{v \in V(H)} \frac{1}{\langle \rho(v), e_1 \rangle^2}$ as well as the **Lovaz theta function**

$\vartheta(H) = inf\{\vartheta(H, \rho) : \rho \text{ an orthogonal representation}\}$

Interesting side note: The Lovas Theta function can be approximated in polynomial time

Lemmas about $\vartheta(H, \rho)$

For any Graph H and orthogonal representation ρ we have:

Lemma A: $\alpha(H) \leq \vartheta(H, \rho)$

Proof idea: For a system of orthonormal vectors $(v_1, \ldots, v_m)$ in $\mathbb{R}^n$ and an arbitrary vector u we have: $\sum_{i=1}^{m} \langle v_i, u \rangle^2 \leq \|u\|^2$. Plug in $u = e_1$ and orthonormal system $(\rho(v), v \in I)$ where I is an independent set in H

Lemma B: For two Graphs H_1, H_2 and the strong product $H_1 \cdot H_2$ we have $\vartheta(H_1 \cdot H_2, \rho) \leq \vartheta(H_1, \rho_1) \cdot \vartheta(H_2, \rho_2)$

Proof idea: if ρ_1, ρ_2 are orthogonal representations of H_1, H_2 then $\rho := \rho_1 \otimes \rho_2$ is an orthogonal representation of $H_1 \cdot H_2$

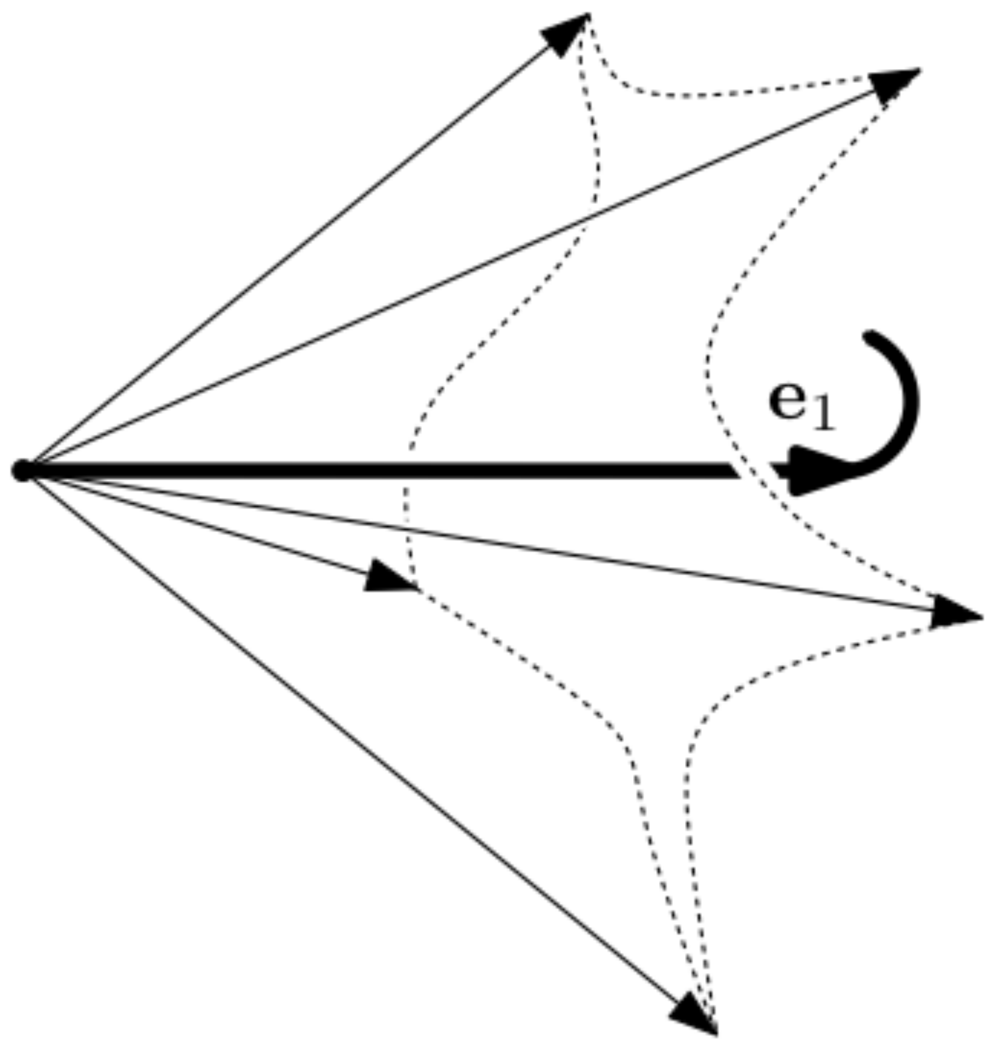
Putting the two thing together we get $\alpha(H^k) \leq \vartheta(H, \rho)^k$ and in particular $\Theta(H) \leq \vartheta(H) \leq \vartheta(H, \rho)$

Back to $\Theta(C_5)$

We now give a orthogonal representation ρ_{LU} called the Lovász umbrella of C_5

Imagine a closed umbrella with 5 ribs an the tube e_1, and open the ribs until non neighboring ribs become orthogonal.

If we let the ribs $v_1, \ldots, v_5$ be unit vectors and assign the i-th node of G to v_i we get the orthogonal representation ρ_{LU}



we can now calculate the opening angle of the obtain $\langle v_i, e_1 \rangle = 5^{-\frac{1}{4}}$ and therefore:

$$\Theta(C_5) \leq \vartheta(C_5, \rho_{LU}) = max_{v \in V(H)} \frac{1}{\langle \rho(v), e_1 \rangle^2} = \sqrt{5}$$

so we now know $\Theta(C_5) = \sqrt{5}$

Outlook

We currently have $\alpha(G^k)^{\frac{1}{k}} \leq \Theta(G) \leq \vartheta(G)$. Do we have $\Theta(G) = \vartheta(G)$? No.

Unfortunately for many graphs $\Theta(G)$ is not known and one does not need to look far as for example $\Theta(C_7)$ is unknown. In fact for all odd cycles the Shannon capacity is unknown.

The one class where it is very easy to determine the Shannon capacity are so called perfect graphs. Which are graphs G such that for any subgraph $H \subseteq G$ we have $\chi(H) = \omega(H)$ where $\chi(H)$ is the minimum size of a coloring of H and $\omega(H)$ is the maximum size of a clique. (Most notably bipartite graphs)

For the disjoint union of two graphs denoted $G + H$ we have:
$\Theta(G + H) \geq \Theta(G) + \Theta(H)$ does equality hold? No.

Further investigation of $\vartheta(G)$

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Let G be a graph on the vertices $\{1,\ldots,n\}$ then $\vartheta(G)$ is the largest eigenvalue of $A = (a_{i,j})_{i,j=1}^{n}$ where $a_{i,j} = 1$ if $i = j$ or i and j are nonadjacent.

This connection tho the eigenvalues of a matrix allows us to prove many more interesting results which lead to the following result:

For n odd we have: $\vartheta(C_n) = \dfrac{n\cos(\frac{\pi}{n})}{1+\cos(\frac{\pi}{n})}$