The Kissing Number Problem

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1 Introduction

The Kissing Number Problem (KNP) can be thought of as:

For a convex set K how many non-overlapping translated copies of K can touch K simultaneously?

We will mainly be focusing on the case where K = $\mathbb{S}^{d-1}$. The following are some useful definitions that will be used later on.

Lattice: Let $A = (a_i)_{i\in[n]}$ be n linearly independent vectors in $\mathbb{R}^d$ then $\Lambda := \{\sum_{i=1}^{n} z_i a_i | \forall i \in [n] : z_i \in \mathbb{Z}\} \subset \mathbb{R}^d$ is the **Lattice** generated by A.

Packing: Let X be a discrete set in $\mathbb{R}^d$ and K a convex set in $\mathbb{R}^d$ then $K + X := \{K + x | x \in X$ and none of the copies of K overlap with each other$\}$ is called the **translative packing** of K. If X is a lattice then $K + X$ is a **lattice packing** of K.

Kissing Configuration: A kissing configuration is a translative packing of K such that all copies of K touch K, without overlapping.

Kissing Number: Let $k(K)$ denote the kissing number of K, i.e. the maximal size of a kissing configuration, where the positions of the copies of K need not have any particular order or regularity, while $k^*(K)$ denotes the kissing number when the copies of K must be arranged in a lattice. $k(n)$ will denote $k(\mathbb{S}^{n-1})$, likewise for $k^*(n)$.

Integral Lattice: A lattice Λ is an **integral lattice** if $\forall u \in \Lambda : (u, u) \in \mathbb{Z}$, and an **even integral lattice** if $(u, u) \in 2\mathbb{Z}$, and an **odd integral lattice** otherwise.

Two useful results are, $k^*(2) = k(2) = 6$, the proof of which is easy enough and will be presented during the presentation, and that by definition $k^* \leq k$.

2 General Statement of the KNP and some Results

As we saw in section 1, k and k^* are the largest number of identical copies of a convex set P obtained by arbitrary translations, or by a lattice arrangement of P, respectively, that are non-overlapping while still touching the original set. A nice first upper bound on k and k^* is attributed to Minkowski and Hadwiger.
Theorem 2: (Minkowski-Hadwiger[1])

For an n-dimensional convex body K, $k^*(K) \leq k(K) \leq 3^n - 1$, where equality holds for parallelepipeds.

The proof consists of a reduction to the case where K is centrally symmetric; showing that if a translate $K + x$ of K touches K it is contained in $3K$; finally making use of the n dimensional volume being homogeneous of degree n. This proof nicely demonstrates how the KNP is a localized version of the packing problem. For a detailed proof see [1] Ch. 1.1.

If we now return our focus to spheres, we can use a geometric interpretation of the scalar product,

$$(a, b) = \|a\|\|b\| \cos\theta, \tag{1}$$

where θ is that angle between the vectors a and b, to reformulate the question behind k and k^*. Instead of asking for the maximal number of non-overlapping congruent copies of a sphere, that touch the original sphere, we can ask,

What is the largest set of $x_i \in \mathbb{S}^{d-1}$ such that $(x_i, x_j) \leq \frac{1}{2}, \forall i \neq j$?

Here the x_i are referred to as the kissing points, i.e. where the copies of $\mathbb{S}^{d-1}$ touch the original $\mathbb{S}^{d-1}$. Our new requirement is equivalent to demanding that the minimal angle between any two kissing points is $\frac{\pi}{3}$. This is the same as trying to find the maximal spherical code with distance $\frac{\pi}{3}$ in any given dimension. The new formulation is in fact equivalent to the original question, which can be seen by recalling our result for $d = 2$, $k(2) = 6$, and that any plane spanned by x_i and x_j contains the origin and hence a copy of $\mathbb{S}^1$.

A general upper bound for d dimensions was found by Kabatiansky and Levenshtein, as stated in [3], who showed:

$$k(d) \leq 2^{0.401d(1+o(1))} \tag{2}$$

Dimension	Lower bound	Upper bound
1	2	
2	6	
3	12	
4	24[7]	
5	40	44
6	72	78
7	126	134
8	240	
9	306	364
10	500	554

This table [4] shows some known results in small dimensions.

3 The Gregory Newton Problem

The Gregory Newton Problem (GNP) ask the question of what is $k(3)$. The name of the problem comes from a disagreement on this matter by Sir Isaac Newton and David Gregory on the precise value of $k(3)$. Newton believed it equalled 12, while Gregory claimed it equalled 13.
That $k^*(3) \geq 12$ can be seen by looking at the lattice, Λ_3, generated by $\{(2,0,0),(1,\sqrt{3},0)\,(1,\frac{1}{\sqrt{3}},\frac{2\sqrt{6}}{3})\}$.
<u>Theorem</u> 1: (Hoppe, Schütte and van der Waerden, and Leech[1])

$$k^*(3) = k(3) = 12 \tag{3}$$

To prove this, we require 3 main steps or ideas. Firstly, we do some rudimentary computations in geodesic geometry. Secondly, we require Euler's Formula for polygons: $v - e + f = 2$, where v corresponds to the number of corners or vertices, e corresponds to the number of edges, and f to the number of faces. Finally, we create a contradiction for the existence of a quadrilateral arrangement on $\mathbb{S}^2$.
<u>Proof:</u> Let $X = \{x_1, ..., x_n\} \subset \mathbb{S}^2$ such that $\mathbb{S}^2 + (2X \cup \{0\})$ is a kissing configuration. Let us denote by $\|x_i, x_j\|_g$ the geodesic distance between $x_i, x_j \in X$. We will construct a "planar" graph on the surface of $\mathbb{S}^2$ where X corresponds

to the set of vertices, and $(x_i, x_j) \in E$ if $\|x_i, x_j\|_g < \arccos(\frac{1}{7})$, with this construction we obtain a "planar" graph on or polygonal covering of $\mathbb{S}^2$. With some computations on geodesic geometry, we see that the angle between any two edges must be greater than $\frac{\pi}{3}$, and hence, any vertex x_i has at most 5 neighbours. Further computations yield that

$$s(P_3) > 0.5512, s(P_4) > 1.3388, s(P_5) > 2.2261 \tag{4}$$

and that the lower bound is strictly increasing with n, given our minimal edge length.
Euler's formula now yields:

$$2v - 4 = 2e - 2f = 2e - 2\sum_{i\geq 3} f_i \tag{5}$$

$$= \sum_{i\geq 1} i f_{i+2} \tag{6}$$

This in turn yields the inequality,

$$s(\mathbb{S}^2) = 4\pi > 0.5512 f_3 + 1.3388 f_4 + 2.2261(f_5 + ...) \tag{7}$$

$$> 0.5512 \sum_{i\geq 1} i f_{i+2} + 0.2314 f_4 + 0.57(f_5 + ...) \tag{8}$$

Which implies, $v \geq 13$. However by our previous calculation we get

$$0.44 \geq 0.2314 f_4 + 0.57(f_5 + ...), \tag{9}$$

therefore $f_4 \in \{0, 1\}$ and $f_n = 0$ for $n \geq 5$. If $f_4 = 1$ by Euler's formula we get $f_3 = 20$ and $e = 32$, a quick construction yields that such a graph cannot exist. If $f_4 = 0$, we have $f_3 = \frac{2e}{3} \Rightarrow 13 + \frac{2e}{3} = e + 2 \Rightarrow e = 33 \Rightarrow \overline{d} > 5$, this however contradicts our previous result of $d(x) \leq 5$. Hence $v < 13$ and $k^*(3) = 12$ $\square$

4 The KNP in eight and 24 dimensions

In this section we will discuss two results accredited to Levenshtein, Odlyzko, and Sloane ([1] Ch.9). Both proofs proceed in a similar manner and make use of Delsarte's Lemma ([1] Ch.8), which gives conditions on a polynomial function $f(t) = \sum_{i=0}^{k} f_i P_i^{\alpha,\alpha}(t)$, where $P_i^{\alpha,\alpha}$ is the Jacobi polynomial, for $\alpha = \frac{n-3}{2}$, relating to a spherical $\{n, m, \phi\}$ code, such that maximal choice of m, given n and ϕ, is bounded by $\frac{f(1)}{f_0}$, where $k^*(d) = m[d, \frac{\pi}{3}]$.

But first let us introduce some useful notions, that of the radius of a lattice and the set of minimal vectors in Λ:

$$r_\Lambda = \min\{\frac{1}{2}\|u\| | u \in \Lambda \backslash \{0\}\} \tag{10}$$

$$M(\Lambda) = \{u \in \Lambda | \|u\| = 2r\} \tag{11}$$

It is easily seen that for Λ a d dimensional lattice, $r\mathbb{S}^{d-1} + \Lambda$ is a lattice Packing of $r\mathbb{S}^{d-1}$ and for any $u \in M(\Lambda)$, $r\mathbb{S}^{d-1} + u$ touches $r\mathbb{S}^{d-1}$ only in its boundary, and hence $k^*(d) = \max_{r_\Lambda = 1} card(M(\Lambda))$.
Now using the E_8 and Leech lattices $E_8 = \{u \in \frac{1}{2}\mathbb{Z}_8 | u_i - u_j \in \mathbb{Z} \;\&\; \sum_{i=1}^{8} u_i \in 2\mathbb{Z}\}$, Λ_{24}, we obtain lower bounds on $k^*(8)$, and $k^*(24)$, with $M(E_8) = 240$, and $M(\Lambda_{24}) = 196560$, respectively. Now "all" that remains, is to find a polynomial satisfying Delsarte's conditions that gives a tight upper bound. This is exactly what was done, yielding the polynomials,

$$f_8(t) = P_0(t) + \frac{16}{7}P_1 + \frac{200}{63}P_2 + \frac{832}{63}P_3 + \frac{1216}{429}P_4 + \frac{5120}{3003}P_5 + \frac{2560}{4641}P_6 \tag{12}$$

and

$$f_{24}(t) = P_0 + f_1P_1 + f_2P_2 + f_3P_3 + f_4P_4 + f_5P_5 + f_6P_6 + f_7P_7 + f_8P_8 + f_9P_9. \tag{13}$$

Both offering tight upper bounds and thereby giving the results, $k^*(8) = 240$, and $k^*(24) = 196560$, respectively.

5 Uniqueness of the Kissing Configuration in dimension 8

Theorem 3: (Bannai-Sloane [1])

There is a unique way, up to isometry, of arranging 240 non-overlapping unit spheres in $\mathbb{E}^8$ so that they all touch another common unit sphere.

The proof of this theorem shows first that any such configuration is a $\{8, 240, \frac{\pi}{3}\}$ spherical code. Then by an optimization argument, shows such a code leads to an integral lattice with "short" generating vectors, and then we apply a lemma, attributed to Kneser [1], which states, any integral lattice generated by vectors of length 1 or $\sqrt{2}$ can be written as a sum of the lattices A_n, Z_n, $n \geq 1$, D_n,$n \geq 4$, and E_n, $n \in \{6, 7, 8\}$, in order to conclude that the only such sum is equal to exactly E_8.
So let us take any $\{8, 240, \frac{\pi}{3}\}$ code, $\aleph^* = \{u_1^*, ..., u_{240}^*\}$. Such a code can be

rewritten as an optimal solution of a linear programming problem. Such optimization problems have dual problems, so let the polynomial $f_8(t)$ be the solution to the dual problem, we note that this polynomial satisfies certain qualities, that imply the only possible values for inner products in $\aleph^*$ are $0, \pm\frac{1}{2}, \pm 1$. Then taking the lattice, $\Lambda^* = \{\sum_{i=1}^{240} \sqrt{2} z_i u_i^* : z_i \in \mathbb{Z}\}$, we obtain $M(\Lambda^*) = \sqrt{2}\aleph^*$ and that this lattice is an even integral lattice.
Finally noting that since E_8 is the only such lattice admitted by Kneser's lemma with at least 240 minimal vectors, Λ^* must be isometric to E_8, and hence $M(\Lambda^*)$ is isometric to $M(E_8)$.
There exists an identical theorem for the arrangement in 24 dimensions, also by Bannai and Sloane, the proof of which is a bit more extensive and can be found in [1] Ch. 9.4.

6 Musin's Theorem

This section will outline the idea of the proof given by Oleg R. Musin in [2], in which he shows $k(4) = 24$.
In his paper, Musin finds a maximal spherical $\frac{\pi}{3}$-code X in $\mathbb{S}^3$, i.e. a set X such that $\forall x, y \in X : (x, y) \leq \cos\frac{\pi}{3} = \frac{1}{2}$. By extending Delsarte's method, Musin obtained a strict upper bound $k(4) < 25$, and since $k(4) \geq 24$ is known, by taking the arrangement given by the D_4 lattice, the claim follows.

The proof is further broken down into showing two lemmas, Lemma A and Lemma B, that together yield the claim. To obtain these lemmas, Musin introduces the following polynomial of ninth degree:

$$f_4(t) := \frac{1344}{25}t^9 - \frac{2688}{25}t^7 + \frac{1764}{25}t^5 + \frac{2048}{125}t^4 - \frac{1229}{125}t^3 - \frac{217}{500}t - \frac{2}{125}. \quad (14)$$

Lemma A: Let $X = \{x_1, ..., x_M\}$ be points of the unit sphere $\mathbb{S}^3$. Then

$$S(X) = \sum_{i=1}^{M}\sum_{j=1}^{M} f_4(x_i x_j) \geq M^2. \quad (15)$$

Lemma B: Suppose $X = \{x_1, ..., x_M\}$ is a subset of $\mathbb{S}^3$ such that the angular separation between any two points x_i, x_j is at least $\frac{\pi}{3}$. Then

$$S(X) = \sum_{i=1}^{M}\sum_{j=1}^{M} f_4(x_i x_j) < 25M. \quad (16)$$

Clearly $k(4) = 24$ follows from these two lemmas by letting X be a kissing configuration of $\mathbb{S}^3$ and taking $M = k(4)$.

7 Notation

Here is a brief overview some of the notation used in this handout:
n,d will generally denote natural numbers
$[n] := \{1, 2, 3, ..., n-1, n\}$
I_d shall denote the d dimensional unit cube.
$s(K)$ shall denote the surface area of a set K.
P_n shall refer to a polygon with n sides.
$(,)$ shall be the euclidean scalar product.

References

[1] Chuanming Zong. *Sphere Packings*. Springer-Verlag, 1999. ISBN: 0-387-98794-0.

[2] Oleg R. Musin. '"The kissing number in four dimensions'. In: *Annals of Mathematics* 168 (2008), pp. 1–32.

[3] N. J. Sloane J. H. Conway. *Sphere Packings, Lattices and Groups*. 3rd edition. Springer-Verlag, 2017.

[4] URL: `https://en.wikipedia.org/wiki/Kissing_number#Some_known_bounds`. (accessed: 26.03.2021).