The Kissing Number Problem

Ben O'Sullivan

1 Introduction

The Kissing Number Problem (KNP) can be thought of as:

For a convex set $K$ how many non-overlapping translated copies of $K$ can touch $K$ simultaneously?

We will mainly be focusing on the case where $K = \mathbb{S}^{d-1}$. The following are some useful definitions that will be used later on.

Lattice: Let $A = (a_i)_{i \in [n]}$ be $n$ linearly independent vectors in $\mathbb{R}^d$ then $\Lambda := \{ \sum_{i=1}^{n} z_i a_i | \forall i \in [n] : z_i \in \mathbb{Z} \} \subset \mathbb{R}^d$ is the Lattice generated by $A$.

Packing: Let $X$ be a discrete set in $\mathbb{R}^d$ and $K$ a convex set in $\mathbb{R}^d$ then $K + X := \{ K + x | x \in X \}$ is called the translative packing of $K$. If $X$ is a lattice then $K + X$ is a lattice packing of $K$.

Kissing Configuration: A kissing configuration is a translative packing of $K$ such that all copies of $K$ touch $K$, without overlapping.

Kissing Number: Let $k(K)$ denote the kissing number of $K$, i.e. the maximal size of a kissing configuration, where the positions of the copies of $K$ need not have any particular order or regularity, while $k^*(K)$ denotes the kissing number when the copies of $K$ must be arranged in a lattice. $k(n)$ will denote $k(\mathbb{S}^{n-1})$, likewise for $k^*(n)$.

Integral Lattice: A lattice $\Lambda$ is an integral lattice if $\forall u \in \Lambda : (u,u) \in \mathbb{Z}$, and an even integral lattice if $(u,u) \in 2\mathbb{Z}$, and an odd integral lattice otherwise.

Two useful results are, $k^*(2) = k(2) = 6$, the proof of which is easy enough and will be presented during the presentation, and that by definition $k^* \leq k$.\
2 General Statement of the KNP and some Results

As we saw in section 1, $k$ and $k^*$ are the largest number of identical copies of a convex set $P$ obtained by arbitrary translations, or by a lattice arrangement of $P$, respectively, that are non-overlapping while still touching the original set. A nice first upper bound on $k$ and $k^*$ is attributed to Minkowski and Hadwiger.

**Theorem 2: (Minkowski-Hadwiger[1])**

For an $n$-dimensional convex body $K$, $k^*(K) \leq k(K) \leq 3^n - 1$, where equality holds for parallelepipeds.

The proof consists of a reduction to the case where $K$ is centrally symmetric; showing that if a translate $K + x$ of $K$ touches $K$ it is contained in $3K$; finally making use of the $n$ dimensional volume being homogeneous of degree $n$. This proof nicely demonstrates how the KNP is a localized version of the packing problem. For a detailed proof see [1] Ch. 1.1.

If we now return our focus to spheres, we can use a geometric interpretation of the scalar product,

\[(a, b) = \|a\|\|b\| \cos \theta,\]

where $\theta$ is that angle between the vectors $a$ and $b$, to reformulate the question behind $k$ and $k^*$. Instead of asking for the maximal number of non-overlapping congruent copies of a sphere, that touch the original sphere, we can ask,

**What is the largest set of $x_i \in S^{d-1}$ such that $(x_i, x_j) \leq \frac{1}{2}$, $\forall i \neq j$?**

Here the $x_i$ are referred to as the kissing points, i.e. where the copies of $S^{d-1}$ touch the original $S^{d-1}$. Our new requirement is equivalent to demanding that the minimal angle between any two kissing points is $\frac{\pi}{3}$. This is the same as trying to find the maximal spherical code with distance $\frac{\pi}{3}$ in any given dimension. The new formulation is in fact equivalent to the original question, which can be seen by recalling our result for $d = 2$, $k(2) = 6$, and that any plane spanned by $x_i$ and $x_j$ contains the origin and hence a copy of $S^1$.

A general upper bound for $d$ dimensions was found by Kabatiansky and Levenshtein, as stated in [3], who showed:

\[k(d) \leq 2^{0.401d(1+o(1))}\] (2)
This table \[4\] shows some known results in small dimensions.

### The Gregory Newton Problem

The Gregory Newton Problem (GNP) ask the question of what is \(k(3)\). The name of the problem comes from a disagreement on this matter by Sir Isaac Newton and David Gregory on the precise value of \(k(3)\). Newton believed it equalled 12, while Gregory claimed it equalled 13. That \(k^*(3) \geq 12\) can be seen by looking at the lattice, \(\Lambda_3\), generated by \(\{(2, 0, 0), (1, \sqrt{3}, 0), (1, \frac{1}{\sqrt{3}}, \frac{2\sqrt{6}}{3})\}\).

**Theorem 1:** (Hoppe, Schütte and van der Waerden, and Leech\[1\])

\[
k^*(3) = k(3) = 12
\]

To prove this, we require 3 main steps or ideas. Firstly, we do some rudimentary computations in geodesic geometry. Secondly, we require Euler’s Formula for polygons: \(v - e + f = 2\), where \(v\) corresponds to the number of corners or vertices, \(e\) corresponds to the number of edges, and \(f\) to the number of faces. Finally, we create a contradiction for the existence of a quadrilateral arrangement on \(S^2\).

**Proof:** Let \(X = \{x_1, \ldots, x_n\} \subset S^2\) such that \(S^2 + (2X \cup \{0\})\) is a kissing configuration. Let us denote by \(\|x_i, x_j\|_g\) the geodesic distance between \(x_i, x_j \in X\). We will construct a ”planar” graph on the surface of \(S^2\) where \(X\) corresponds
to the set of vertices, and \((x_i, x_j) \in E\) if \(\|x_i, x_j\|_g < \arccos(\frac{1}{7})\), with this construction we obtain a "planar" graph on or polygonal covering of \(S^2\). With some computations on geodesic geometry, we see that the angle between any two edges must be greater than \(\frac{\pi}{3}\), and hence, any vertex \(x_i\) has at most 5 neighbours. Further computations yield that

\[
s(P_3) > 0.5512, s(P_4) > 1.3388, s(P_5) > 2.2261 \tag{4}\]

and that the lower bound is strictly increasing with \(n\), given our minimal edge length.

Euler's formula now yields:

\[
2v - 4 = 2e - 2f = 2e - 2 \sum_{i \geq 3} f_i = \sum_{i \geq 1} if_{i+2} \tag{5}
\]

This in turn yields the inequality:

\[
s(S^2) = 4\pi > 0.5512 f_3 + 1.3388 f_4 + 2.2261 (f_5 + ...) > 0.5512 \sum_{i \geq 1} if_{i+2} + 0.2314 f_4 + 0.57 (f_5 + ...) \tag{6}
\]

Which implies, \(v \geq 13\). However by our previous calculation we get

\[
0.44 \geq 0.2314 f_4 + 0.57 (f_5 + ...), \tag{7}
\]

therefore \(f_4 \in \{0, 1\}\) and \(f_n = 0\) for \(n \geq 5\). If \(f_4 = 1\) by Euler’s formula we get \(f_3 = 20\) and \(e = 32\), a quick construction yields that such a graph cannot exist. If \(f_4 = 0\), we have \(f_3 = \frac{2e}{3} \Rightarrow 13 + \frac{2e}{3} = e + 2 \Rightarrow e = 33 \Rightarrow d > 5\), this however contradicts our previous result of \(d(x) \leq 5\). Hence \(v < 13\) and \(k^*(3) = 12\)

\[
4 \quad \text{The KNP in eight and 24 dimensions}
\]

In this section we will discuss two results accredited to Levenshtein, Odlyzko, and Sloane ([1] Ch.9). Both proofs proceed in a similar manner and make use of Delsarte’s Lemma ([1] Ch.8), which gives conditions on a polynomial function \(f(t) = \sum_{i=0}^{k} f_i P_i^{\alpha,\alpha}(t)\), where \(P_i^{\alpha,\alpha}\) is the Jacobi polynomial, for \(\alpha = \frac{n-3}{2}\), relating to a spherical \(\{n, m, \phi\}\) code, such that maximal choice of \(m\), given \(n\) and \(\phi\), is bounded by \(\frac{f(\phi)}{f_0}\), where \(k^*(d) = m[d, \frac{\pi}{3}]\).
But first let us introduce some useful notions, that of the radius of a lattice and the set of minimal vectors in $\Lambda$:

$$r_\Lambda = \min\left\{\frac{1}{2}\|u\| : u \in \Lambda \setminus \{0\}\right\} \quad (10)$$

$$M(\Lambda) = \{u \in \Lambda \|u\| = 2r\} \quad (11)$$

It is easily seen that for $\Lambda$ a $d$ dimensional lattice, $rS^{d-1} + \Lambda$ is a lattice Packing of $rS^{d-1}$ and for any $u \in M(\Lambda)$, $rS^{d-1} + u$ touches $rS^{d-1}$ only in its boundary, and hence $k^*(d) = \max_{r_\Lambda = 1} \text{card}(M(\Lambda))$.

Now using the $E_8$ and Leech lattices $E_8 = \{u \in \frac{1}{2}Z_8 | u_i \in Z \& \sum_{i=1}^{8} u_i \in 2Z\}$, $\Lambda_{24}$, we obtain lower bounds on $k^*(8)$, and $k^*(24)$, with $M(E_8) = 240$, and $M(\Lambda_{24}) = 196560$, respectively. Now “all” that remains, is to find a polynomial satisfying Delsarte’s conditions that gives a tight upper bound. This is exactly what was done, yielding the polynomials,

$$f_8(t) = P_0(t) + \frac{16}{7}P_1 + \frac{200}{63}P_2 + \frac{832}{63}P_3 + \frac{1216}{429}P_4 + \frac{5120}{3003}P_5 + \frac{2560}{4641}P_6 \quad (12)$$

and

$$f_{24}(t) = P_0 + f_1P_1 + f_2P_2 + f_3P_3 + f_4P_4 + f_5P_5 + f_6P_6 + f_7P_7 + f_8P_8 + f_9P_9 \quad (13)$$

Both offering tight upper bounds and thereby giving the results, $k^*(8) = 240$, and $k^*(24) = 196560$, respectively.

5 Uniqueness of the Kissing Configuration in dimension 8

Theorem 3: (Bannai-Sloane [1])

There is a unique way, up to isometry, of arranging 240 non-overlapping unit spheres in $\mathbb{E}^8$ so that they all touch another common unit sphere.

The proof of this theorem shows first that any such configuration is a $\{8, 240, \frac{\pi}{3}\}$ spherical code. Then by an optimization argument, shows such a code leads to an integral lattice with "short" generating vectors, and then we apply a lemma, attributed to Kneser [1], which states, any integral lattice generated by vectors of length 1 or $\sqrt{2}$ can be written as a sum of the lattices $A_n$, $Z_n$, $n \geq 1$, $D_n$, $n \geq 4$, and $E_n$, $n \in \{6, 7, 8\}$, in order to conclude that the only such sum is equal to exactly $E_8$.

So let us take any $\{8, 240, \frac{\pi}{3}\}$ code, $N^* = \{u_1^*, ..., u_{240}^*\}$. Such a code can be
rewritten as an optimal solution of a linear programming problem. Such optimization problems have dual problems, so let the polynomial $f_8(t)$ be the solution to the dual problem, we note that this polynomial satisfies certain qualities, that imply the only possible values for inner products in $\mathbb{R}^*$ are $0, \pm \frac{1}{2}, \pm 1$. Then taking the lattice, $\Lambda^* = \{ \sum_{i=1}^{240} \sqrt{2}z_i u_i^* : z_i \in \mathbb{Z} \}$, we obtain $M(\Lambda^*) = \sqrt{2}\mathbb{R}^*$ and that this lattice is an even integral lattice.

Finally noting that since $E_8$ is the only such lattice admitted by Kneser’s lemma with at least 240 minimal vectors, $\Lambda^*$ must be isometric to $E_8$, and hence $M(\Lambda^*)$ is isometric to $M(E_8)$.

There exists an identical theorem for the arrangement in 24 dimensions, also by Bannai and Sloane, the proof of which is a bit more extensive and can be found in [1] Ch. 9.4.

6 Musin’s Theorem

This section will outline the idea of the proof given by Oleg R. Musin in [2], in which he shows $k(4) = 24$.

In his paper, Musin finds a maximal spherical $\frac{\pi}{3}$-code $X$ in $S^3$, i.e. a set $X$ such that $\forall x, y \in X : (x, y) \leq \cos \frac{\pi}{3} = \frac{1}{2}$. By extending Delsarte’s method, Musin obtained a strict upper bound $k(4) < 25$, and since $k(4) \geq 24$ is known, by taking the arrangement given by the $D_4$ lattice, the claim follows.

The proof is further broken down into showing two lemmas, Lemma A and Lemma B, that together yield the claim. To obtain these lemmas, Musin introduces the following polynomial of ninth degree:

$$f_4(t) := \frac{1344}{25} t^9 - \frac{2688}{25} t^7 + \frac{1764}{25} t^5 + \frac{2048}{125} t^4 - \frac{1229}{125} t^3 - \frac{217}{500} t - \frac{2}{125}. \tag{14}$$

**Lemma A:** Let $X = \{x_1, ..., x_M\}$ be points of the unit sphere $S^3$. Then

$$S(X) = \sum_{i=1}^{M} \sum_{j=1}^{M} f_4(x_i x_j) \geq M^2. \tag{15}$$

**Lemma B:** Suppose $X = \{x_1, ..., x_M\}$ is a subset of $S^3$ such that the angular separation between any two points $x_i, x_j$ is at least $\frac{\pi}{3}$. Then

$$S(X) = \sum_{i=1}^{M} \sum_{j=1}^{M} f_4(x_i x_j) < 25M. \tag{16}$$

Clearly $k(4) = 24$ follows from these two lemmas by letting $X$ be a kissing configuration of $S^3$ and taking $M = k(4)$. 

6
7 Notation

Here is a brief overview some of the notation used in this handout:

\( n,d \) will generally denote natural numbers

\[ n \] := \{1,2,3,\ldots,n−1,n\}

\( I_d \) shall denote the \( d \) dimensional unit cube.

\( s(K) \) shall denote the surface area of a set \( K \).

\( P_n \) shall refer to a polygon with \( n \) sides.

\( (,\)\ shall be the euclidean scalar product.

References


