Linear Programming Bounds for Sphere Packings I

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1 Introduction

One of our goals of this series is to understand and bound the maximal sphere packing density in $\mathbb{R}^n$. We have seen solutions for low dimensions but as soon as we start to consider higher dimensions our understanding rapidly diminishes. To combat that problem, the main goal today will be to construct an upper bound on that density for large dimensions. The idea will be to use the spherical codes from two weeks ago [1]. In fact, we will use an improved upper bound on the size of those and a neat geometrical argument to transform this upper bound to one suitable for the sphere packing density. On the way of doing so we will additionally see that the results we encounter give us an upper bound on the kissing number problem for spheres which we have seen last week [2].

2 Linear Programming Bounds for Spherical Codes

Recall from two weeks ago,

Definition 1 (Spherical Code)**.** *A spherical code of dimension n, cardinality m and minimum angle φ is a set of m points $X \subset \mathrm{S}^{n-1}$ such that for all $x, y \in X$ with $x \neq y$,*

$$\langle x, y \rangle \leq \cos \varphi.$$

Furthermore, denote by $m[n, \varphi]$ the maximal size, i.e. m, of any spherical code with dimension n and minimum angle φ.

Note, [Def. 1] corresponds to the definition we have seen two weeks ago [1] for $A = [-1, \cos \varphi]$.

Remark 1. *As we have seen last week [2] the special case where $\varphi = \frac{\pi}{3}$ corresponds to the kissing configuration of the unit sphere and hence $m[n, \frac{\pi}{3}]$ is its kissing number.*

As already mentioned, we want to improve the upper bound on $m[n, \varphi]$. Before doing so, let us recall the bound we want to improve upon. Note, the notation used here is not identical to the one in [1]. For example, we use a simplified version of the Jacobi polynomials instead of the Gegenbauer polynomials.

Definition 2 (Jacobi Polynomials - Simplified)**.** *Let $\alpha > -1$ and $k \in \mathbb{N}_0$ then the simplified Jacobi polynomial P_k^α is given by*

$$P_k^\alpha(t) := \frac{1}{2^k}\sum_{i=0}^{k}\binom{k+\alpha}{i}\binom{k+\alpha}{k-i}(t+1)^i(t-1)^{k-i}.$$

Denote by $t_1(k,\alpha) \geq t_2(k,\alpha) \geq \ldots \geq t_k(k,\alpha)$ the k zeros of the polynomial $P_k^\alpha(t)$.

Definition 3. *Let $\alpha = \frac{n-3}{2}$. Denote by $\mathcal{F}(n,\varphi)$ the set of polynomials of the following form.*

$$f(t) = \sum_{i=0}^{k} f_i P_i^\alpha(t),$$

where $f_0 > 0$, $\forall i \in \mathbb{N}$ $f_i \geq 0$ and $f(t)$ is compatible with $A = [-1, \cos\varphi]$, i.e.

$$f\big(A\big) \subseteq (-\infty, 0].$$

Theorem 1 (Delsarte)**.** *Choose any $f \in \mathcal{F}(n,\varphi)$, then*

$$m[n,\varphi] \leq \frac{f(1)}{f_0}.$$

(Proof: [3, 1])

3 Kabatyanski-Levenshtein Bound for Spherical Codes

Note that the bound in [Thm. 1] depends on our choice of $f \in \mathcal{F}(n,\varphi)$. We can use that fact to our advantage and define

$$I(n,\varphi) := \inf_{f \in \mathcal{F}(n,\varphi)} \frac{f(1)}{f_0}.$$

This allows us to easily improve the upper bound to

$$m[n,\varphi] \leq I(n,\varphi).$$

The advantage of this bound is that it solely depends on n and φ. The problem with it is that it is harder to calculate. Nevertheless, the following theorem manages to bound it from above and thus bounds $m[n,\varphi]$ as well.

Theorem 2 (Kabatyanski-Levenshtein)**.** *Assume $0 < \varphi < \frac{\pi}{2}$ and denote,*

$$c := \frac{1-\sin\varphi}{2\sin\varphi}.$$

Then, for sufficiently large n,

$$\frac{\log m[n,\varphi]}{n} \leq \frac{\log I(n,\varphi)}{n} \ll (c+1)\log(c+1) - c\log c.$$

Remark 2. *We have seen last week [2] that*

$$k(n) \leq 2^{0.401\, n\big(1+o(1)\big)}.$$

That is a direct consequence of [Thm. 2] for $\varphi = \frac{\pi}{3}$, following [Rem. 1].

In order to proof [Thm. 2] we need two additional results. Both of which are ultimately just properties of the Jacobi polynomials.

Lemma 1. *Let $s := \cos\varphi$ and*

$$\tau := -\frac{k+1}{k+\alpha+1}\frac{P^{\alpha}_{k+1}(s)}{P^{\alpha}_{k}(s)}.$$

If $t_1(k,\alpha) < s < t_1(k+1,\alpha)$, then

$$I(n,\varphi) \leq \frac{(1+\tau)^2}{(1-s)\tau}\binom{k+2\alpha+1}{k}.$$

And if $s - t_1(k,\alpha)$ is sufficiently small or negative, then

$$I(n,\varphi) \leq \frac{4}{1-t_1(k+1,\alpha)}\binom{k+2\alpha+1}{k}.$$

Proof: The full proof is in [3]. The idea is to construct a polynomial $f \in \mathcal{F}(n,\varphi)$ such that $\frac{f(1)}{f_0}$ is off the right form. The idea of the second part is to use the fact that $I(n,\varphi)$ is an increasing function of s and to consider the first part in the special case where $\tau = 1$.

Lemma 2. *Let $c > 0$ and $k(n)$ such that*

$$\lim_{n\to\infty}\frac{\alpha}{k} = \lim_{n\to\infty}\frac{n-3}{2k} = \frac{1}{2c},$$

then

$$\lim_{k\to\infty} t_1(k,\alpha) = \frac{2\sqrt{c(1+c)}}{1+2c}.$$

Proof: The full proof is again in [3]. The idea is to bound the series from both sides by a series that converges to the right value. The bounding series can be obtained by considering a modified version of the Jacobi polynomial with equal zeros and an ODE that this modification solves.

Proof of [Thm. 2]. Recall,

$$c = \frac{1-\sin\varphi}{2\sin\varphi} > 0.$$

Using the right $k(n)$, [Lem. 2] gives us

$$\lim_{k\to\infty} t_1(k,\alpha) = \frac{2\sqrt{c(1+c)}}{1+2c} = \cos\varphi.$$

Additionally, for k large enough we have by [Lem. 1] and the fact that $t_1(k,\alpha)$ is increasing, that

$$\begin{aligned} I(n,\varphi) &\leq \frac{4}{1-t_1(k+1,\alpha)}\binom{k+2\alpha+1}{k} \\ &\leq \frac{4}{1-\cos\varphi}\binom{k+2\alpha+1}{k}. \end{aligned}$$

Using the fact that $\lim_{k\to\infty}\frac{k}{2\alpha} = c$ and Stirling's formula gives us

$$\frac{\log I(n,\varphi)}{n} \ll (c+1)\log(c+1) - c\log c.$$

This concludes the proof as $m[n,\varphi] \leq I(n,\varphi)$. □

4 Sphere Packing Density and Maximal Size of Spherical Codes

The Kabatyanski-Levenshtein bound from [Thm. 2] has already proven itself to be useful for the kissing number problem. But as promised it can also be used to bound the sphere packing density in $\mathbb{R}^n$. The idea here is to construct from a given sphere packing a corresponding spherical code which has a cardinality corresponding to the density of the packing. This is achieved in the following theorem through a rather neat geometrical argument.

Theorem 3 (Cohn-Zhao)**.** *For all $n \geq 1$ and $\frac{\pi}{3} \leq \varphi \leq \pi$*

$$\Delta_{\mathbb{R}^n} \leq m[n,\varphi]\cdot \sin^n\frac{\varphi}{2}.$$

Proof. Let $\mathcal{P}$ be a sphere packing of unit spheres in $\mathbb{R}^n$ with density Δ. Consider S_R^{n-1}, a $n-1$ dimensional sphere of radius $R \in [1,2]$, in $\mathbb{R}^n$. It can be located such that it contains at least ΔR^n center points of spheres in $\mathcal{P}$ while none of them is concentric with S_R^{n-1}. This is the case as for a uniformly random location

$$\mathbb{E}\left[\#\text{ centerpoints of }\mathcal{P}\text{ in }S_R^{n-1}\right] = \frac{\mathbb{E}\left[\text{Area inside }S_R^{n-1}\cap\mathcal{P}\right]}{\text{Area inside }p\in\mathcal{P}} = \Delta R^n,$$

and so there has to be at least one location at least matching the expected value. Note, that the non-concentricity condition only affects a null set of possible locations.

Having chosen an appropriate location for S_R^{n-1} we can project the center points of the spheres in $\mathcal{P}$ contained within S_R^{n-1} radially onto S_R^{n-1}.

Claim 1. *The projected points are separated by angles of at least φ, where*

$$\sin\frac{\varphi}{2} = \frac{1}{R}.$$

Note, the angles considered are the ones at the center point of S_R^{n-1} between the corresponding radial lines.

This means that the projected points are a spherical code of dimension n, cardinality ΔR^n and minimal angle φ and as such we have

$$\begin{aligned} \Delta R^n &\leq m[n,\varphi] \\ \Delta &\leq m[n,\varphi]\cdot\sin^n\frac{\varphi}{2} \\ \Delta_{\mathbb{R}^n} &\leq m[n,\varphi]\cdot\sin^n\frac{\varphi}{2}. \end{aligned}$$

Note, the restriction $1 \leq R \leq 2$ is translated to $\frac{\pi}{3} \leq \varphi \leq \pi$ by the equation in the claim. It only remains to proof the claim.

Proof of [Claim 1]. Denote by γ the angle in consideration. Firstly, the condition $R \leq 2$ ensures that the projection is injective. Let $u \neq v$ be the projections and denote by s the center point of S_R^{n-1}. Clearly, $\mathrm{d}(s,u) = R = \mathrm{d}(s,v)$ and $x := \mathrm{d}(u,v) \geq 2$. Hence, by the law of cosines

$$\cos\gamma = 1 - \left(\frac{x}{2R}\right)^2.$$

The right hand size is maximised for $x \geq 2$ at $x = 2$ and so

$$\cos\gamma \leq 1 - \frac{1}{R^2} = 1 - \sin^2\frac{\varphi}{2} = \cos\varphi.$$

In conclusion, $\gamma \geq \varphi$, since $\cos(\cdot)$ is strictly decreasing on the relevant section.

□

5 Upper Bound for Sphere Packing Density

To conclude our discussions, here is a table form [4] summarising the upper bounds for the densities of the sphere packing attained through [Thm. 2] and [Thm. 3].

n	$\Delta_{\mathbb{R}^n} \leq \ldots$
12	9.666×10^{-1}
24	2.637×10^{-2}
36	4.951×10^{-4}
48	7.649×10^{-6}
60	1.046×10^{-7}
72	1.322×10^{-9}
84	1.574×10^{-11}
96	1.786×10^{-13}
108	1.942×10^{-15}
120	2.051×10^{-17}
240	1.267×10^{-37}
360	3.003×10^{-58}
480	4.484×10^{-79}
600	5.036×10^{-100}

References

[1] M. Wiedmer, “Spherical codes and designs,” *Handout*, 2021. `https://metaphor.ethz.ch/x/2021/fs/401-3520-21L/sc/SphericalCodesandDesigns.pdf`.

[2] B. O’Sullivan, “The kissing number problem,” *Handout*, 2021. `https://metaphor.ethz.ch/x/2021/fs/401-3520-21L/sc/KNP_Handout.pdf`.

[3] C. Zong, *Sphere Packings*. Springer-Verlag New York, 1999.

[4] H. Cohn, Y. Zhao, *et al.*, “Sphere packing bounds via spherical codes,” *Duke Mathematical J.*, vol. 163, no. 10, pp. 1965–2002, 2014.