Shere Packing, Lattices and Codes Seminar

M. Viazovska's result for sphere packing in dimension 8

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The following statements and exhibition are a summary and paraphrasing of Maryna Viazovska's paper [Via17], where she proves the result we focus on in this talk, and of Henry Cohn's explanatory paper [Coh17].

1 Introduction

On March the 14th 2016, Maryna Viazovska posted to the arXiv a solution to the sphere packing problem in 8 dimensions. In her paper, she proves the following theorem:

Theorem 1.1. *No packing of unit balls in the Euclidean space $\mathbb{R}^8$ has density greater than that of the E_8-lattice packing.*

This is especially remarkable since no proof of optimality had been given for any dimension above 3 and her paper doesn't mention (and in fact doesn't apply to) any dimension between 4 and 7. Within a week, a new result was obtained, using Viazovska's method, showing that the Leech lattice is the densest packing in the Euclidian space $\mathbb{R}^{24}$.

Viazovska's proof relies on the construction of a special function, which enforces the optimality of the E_8 lattice via the Poisson summation formula. Bounds obtained using the *linear programming method* had pushed researchers such as Cohn or Elkies to conjecture the existence of such functions since 2003. However, nobody was close to construct them for any dimension other than 1. Marina's idea resides in the use of modular forms and integral transforms to build such functions.

2 Preliminaries

2.1 Sphere Packing

Let $\mathbb{R}^d$ be the d-dimensional Euclidean real space with distance $\|\cdot\|$ and Lebesgue measure $\mathrm{Vol}(\cdot)$. We denote $B_d(x, r)$ the open ball in $\mathbb{R}^d$ with center $x \in \mathbb{R}^d$ and radius $r > 0$. For a discrete set of points in $\mathbb{R}^d$, noted X, such that $\|x - y\| \geq 2$ for all $x, y \in X$, we call $\mathcal{P} := \bigcup_{x \in X} B_d(x, 1)$ a sphere packing. We recall that the finite density of a packing is given by

$$\Delta_{\mathcal{P}}(r) := \frac{\mathrm{Vol}(\mathcal{P} \cap B_d(x, r))}{\mathrm{Vol}(B_d(x, r))}, \quad r > 0$$

and that the density of the packing $\mathcal{P}$ is then

$$\limsup_{r \to \infty} \Delta_{\mathcal{P}}(r).$$

The number that is of interest to us is the *sphere packing constant* for $\mathbb{R}^d$, i.e. the supremum over all packing densities

$$\Delta_d := \sup_{\substack{\mathcal{P} \subset \mathbb{R}^d \\ \text{sphere packing}}} \Delta_{\mathcal{P}}.$$

Viazovska's paper focuses on the 8-dimensional Euclidean space and proves that the sphere packing corresponding to the E_8-lattice, noted Λ_8, gives the best possible sphere packing density in 8 dimensions.

We recall an easy result about lattice packings:

Lemma 2.1. *Let Λ be a lattice in $\mathbb{R}^d$ with shortest vector length r. Then, this packing has density*

$$\frac{\mathrm{Vol}(B_d(x, r/2))}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)}.$$

2.2 The E_8-lattice

Definition 2.2. The E_8-lattice, denoted Λ_8, is given by

$$\Lambda_8 := \left\{ (x_i) \in \mathbb{Z}^8 + \left(\mathbb{Z} + \frac{1}{2}\right)^8 \mid \sum_{i=1}^{8} x_i \equiv 0 \mod 2 \right\}.$$

Let us list a few properties of Λ_8:

Proposition 2.3. *The E_8-lattice packing in $\mathbb{R}^8$ has density $\pi^4/384 \simeq 0.2536\ldots$.*

Proposition 2.4. *Λ_8 is an even lattice. More precisely, the distances between the points of Λ_8 are of the form $\sqrt{2k}$, for $k \in \mathbb{N}_{>0}$, and in fact all these values are attained.*

Proposition 2.5. *Λ_8 is an unimodular lattice.*

Theorem 2.6. *Every integral unimodular lattice Λ satisfies $\Lambda^* = \Lambda$.*

2.3 The Fourier transform

For an L^1 function $f : \mathbb{R}^d \mapsto \mathbb{R}$, we define the *Fourier transform* of f noted $\hat{f}$ as

$$\hat{f}(y) := \int_{-\infty}^{+\infty} f(x) e^{-2i\pi x \cdot y} dx, \quad y \in \mathbb{R}^d,$$

where $x \cdot y$ denotes the Euclidean scalar product on $\mathbb{R}^d$.

2.4 Linear Programming Bounds

To prove the theorem, Viazovska applies bound from linear programming to a special function. Linear programming asks that this function is *admissible*. To simplify the exhibition, we will take our function to be in a Schwartz space.

Definition 2.7. A function $f : \mathbb{R}^d \mapsto \mathbb{R}$ is said to be a *Schwartz function* if

(1) $f \in C^\infty$;

(2) f decays faster than any inverse power of $\|x\|$ as $\|x\| \to \infty$;

(3) Conditions (1) and (2) hold for any of its derivatives.

To prove Theorem 1.1, Viazovska uses a result from Cohn and Elkies, stating that

Theorem 2.8. *Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a Schwartz function and r a positive real number such that*

(1) $f(0) = \mathcal{F}(f)(0) > 0$;

(2) $\mathcal{F}(f)(y) \geq 0$, $\forall y \in \mathbb{R}^n$;

(3) $f(x) \leq 0$, for $\|x\| \geq r$.

Then, the sphere packing density in $\mathbb{R}^n$ is at most

$$\mathrm{Vol}(B^n_{r/2}) = \frac{\pi^{d/2}}{2^d \Gamma(d/2+1)},$$

where Γ denotes the usual Euler Gamma-function, i.e. the meromorphic continuation of $\Gamma(z) = \int_0^\infty e^s s^z \frac{ds}{s}$, $\Re(z) > 0$ to the whole complex plane.

This theorem tells us how to obtain an upper bound on the sphere packing density, depending on the parameter r, when given any f that satisfies these conditions. The problem of choosing f in order to minimize r remains unsolved for all dimensions other than 1, 8 and 24.

Recall that, in order to obtain the previous theorem, Cohn and Elkies used an important formula.

Theorem 2.9 (Poisson summation formula)**.** *If f is a Schwartz function, then*

$$\sum_{x \in \Lambda} f(x) = \frac{1}{\mathrm{Vol}(\mathbb{R}^n/\Lambda)} \sum_{y \in \Lambda^*} \mathcal{F}(f)(y).$$

3 The Magic Function in Dimension 8

3.1 Heuristics on the magic function

Let f and r be as before and assume that Λ is a lattice in $\mathbb{R}^d$ with minimum vector length r. Then, for the lattice packing $\mathcal{P}_\Lambda$, the Poisson summation formula gives the inequality

$$f(0) \geq \sum_{x \in \Lambda} f(x) = \frac{1}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)} \sum_{y \in \Lambda^*} \mathcal{F}(f)(y) \geq \frac{\mathcal{F}(f)(0)}{\mathrm{Vol}(\mathbb{R}^d/\Lambda)}.$$

Hence, $\Delta_{\mathcal{P}_\Lambda} = \mathrm{Vol}(B^n_{r/2})$ **if and only if** equality holds along the above chain of inequalities, i.e. **if and only if** f and $\mathcal{F}(f)$ vanish on $\Lambda \setminus \{0\}$ and $\Lambda^* \setminus \{0\}$ respectively. This gives us a first idea of what our magic function should, look like. Indeed, we want to

construct a radial Schwartz function that verifies the hypotheses for linear programming and such that f and $\mathcal{F}(f)$ vanish on $\Lambda_8 \setminus \{0\}$.

Note then that these roots should all have order 2 to avoid sign changes, except for the root of f at $\sqrt{2}$ that should have order 1. Follows the depiction of the behaviour of such a function, taken from Henry Cohn's paper [Coh17].

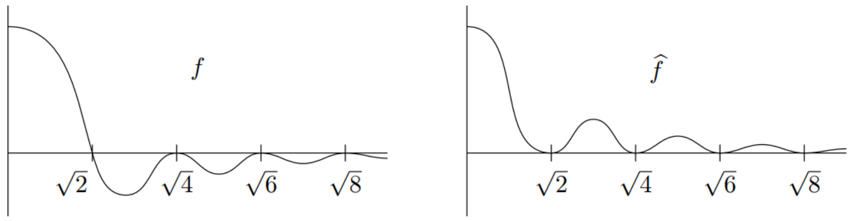


FIGURE 9. A schematic diagram showing the roots of the magic function f and its Fourier transform $\widehat{f}$ in eight dimensions. The figure is not to scale, because the actual functions decrease too rapidly for an accurate plot to be illuminating.

The proof Viazovska gives of Theorem 1.1 consists in showing constructively the existence of such a function, namely, proving the following :

Theorem 3.1. *There exists a radial Schwartz function* $g : \mathbb{R}^8 \mapsto \mathbb{R}$ *that satisfies*

(1) $g(x) \leq 0$ *for all* $\|x\| \geq \sqrt{2}$*;*

(2) $\hat{g}(x) \geq 0$ *for all* $x \in \mathbb{R}^8$*;*

(3) $g(0) = \hat{g}(0) = 1$.

Moreover, $g(x)$ *and* $\hat{g}(x)$ *do not vanish for all vectors* $x \in \mathbb{R}^8$ *with* $\|x\|^2 \notin 2\mathbb{Z}_{>0}$.

4 Building blocks and the link with modular forms

Since we need to control both the function and its Fourier transform, an idea is try to construct the desired Schwartz function as a combination of eigenfunctions of the Fourier transform. Indeed, letting

$$g_+ := \frac{g + \hat{g}}{2}$$
$$g_- := \frac{g - \hat{g}}{2},$$

we obtain $g = g_+ + g_-$, $\mathcal{F}(g_+) = g_+$ and $\mathcal{F}(g_-) = -g_-$ by the Fourier inversion formula. Since we observed before that g and $\mathcal{F}(g)$ must have the same zeroes, the same applies to g_+ and g_-.

We know that the Gaussian function $\varphi(x) = e^{-\pi\|x\|^2}$ is especially well-behaved under the Fourier transform since $\hat{\varphi}(y) = e^{-\pi y^2}$. Hence, it is natural to try to construct the Fourier eigenfunctions with eigenvalues 1 and -1 as continuous linear combinations of Gaussians. In other words, we try to write such functions as

$$g(x) = \int_0^\infty f(t)e^{-t\pi\|x\|^2}dt.$$

This is nothing else than the Laplace transform of some f evaluated at $\pi\|x\|^2$. Now, under some analytic assumptions on f, we can apply the Fourier transform and exchange the order of integration. This yields

$$\begin{aligned}\hat{g}(y) &= \int_0^\infty e^{-\pi\|y\|^2/t}t^{-n/2}f(t)dt \\ &= \int_0^\infty e^{-t\pi\|y\|^2}t^{n/2-2}f(1/t)dt.\end{aligned}$$

Hence, if f were to verify a functional equation such as $f(1/t) = \epsilon t^{2-n/2}f(t)$, for $\epsilon = \pm 1$, we would have $\hat{g} = \epsilon g$. Note that this functional equation looks like that of a modular form on the imaginary axis. Indeed, let $f(t) = \varphi(it)$ for a modular form φ of weight k. Then,

$$f(1/t) = \varphi(-1/it) = (it)^k\varphi(it) = (it)^k f(t).$$

Now, if $k = 2 - n/2$, we obtain

$$f(1/t) = i^{2-n/2}t^{2-n/2}f(t) = \epsilon t^{2-n/2}f(t). \tag{1}$$

Hence, if φ is a modular form that vanishes at $i\infty$ and has no pole on the imaginary axis, f is an eigenfunction of the Fourier transform with eigenvalue i^k. Note that for $n = 8$ we obtain $k = -2$, which makes f a -1-eigenfunction of the Fourier transform.

Note however that this construction doesn't help us at all controlling the roots of g. Recall that we want g to have double zeroes at each vector of Λ_8 of length greater than $\sqrt{2}$ and a single root at each $x \in \Lambda_8 \setminus \{0\}$ of length $\sqrt{2}$ and we want $\mathcal{F}(g)$ to have zeroes of order 2 at all vectors of $\Lambda_8 \setminus \{0\}$. To solve this problem, Viazovska chooses to force the function g to vanish at these points using brute force and multiplying the integral by $\sin\left(\pi\|x\|^2/2\right)^2$.

Now, several new problems arise: $\sin\left(\pi\|x\|^2/2\right)^2$ vanishes with order 4 at 0 whereas don't want it to vanish at the origin and it vanishes with order 2 at $\sqrt{2}$, whereas we only want it to vanish with order 1. To counter this we can try to take an integrand that has a pole of order 4 at 0 and one of order 1 at $\sqrt{2}$.

Additionally, after multiplying by $\sin\left(\pi\|x\|^2/2\right)^2$, it is not clear that our construction of an eigenfunction of the Fourier transform using the modularity of f works anymore.

5 Construction of radial Schwartz functions with double zeroes at lattice points

5.1 Construction of the $+1$ eigenfunction

We define

$$\phi_0 := \frac{(E_4E_2 - E_6)^2}{\Delta} = \frac{(E_4E_2 - E_6)^2}{\frac{1}{1728}(E_4^3 - E_6^2)} = \phi_{-4}E_2^2 + 2\phi_{-2}E_2 + j - 1728, \tag{2}$$

where

$$\phi_{-4} := \frac{E_4^2}{\Delta}, \tag{3}$$

$$\phi_{-2} := \frac{E_2E_4^2 - E_4E_6}{\Delta}, \tag{4}$$

and j is a weakly-holomorphic modular form of weight 0 called the *elliptic j-invariant* that can be defined as

$$j := \frac{1728E_4^3}{E_4^3 - E_6^2}.$$

It follows that ϕ_0 belongs to a family of functions called *weakly holomorphic quasimodular forms of weight 0 and depth 2*. The terms "quasimodular of depth 2" refer to the fact that ϕ_0 can be expressed as a polynomial in E_2, with modular forms as coefficients. Note moreover that Δ doesn't vanish on the upper-half plane, hence ϕ_0 doesn't have any pole on $\mathbb{H}$.

Note that ϕ_0 isn't modular but since E_2 verifies the functional equation

$$z^{-2}E_2\left(\frac{-1}{z}\right) = E_2(z) - \frac{6i}{\pi z}, \tag{5}$$

ϕ_0 transforms as

$$\phi_0\left(\frac{-1}{z}\right) = \phi_0(z) - \frac{12i}{\pi z}\phi_{-2}(z) - \frac{36}{\pi^2}\frac{1}{z^2}\phi_{-4}(z), \tag{6}$$

which implies that

$$g_0(z) := \phi_0\left(\frac{-1}{z}\right)z^2 = \phi_0(z)z^2 - \frac{12i}{\pi}\phi_{-2}(z)z - \frac{36}{\pi^2}\phi_{-4}(z). \tag{7}$$

Moreover, the first few terms of the expansions of ϕ_0, ϕ_{-2}, ϕ_{-4} can be computed. We have

$$\phi_{-4}(z) = q^{-1} + 504 + 73764q + 2695040q^2 + 54755730q^3 + \mathcal{O}(q^4), \tag{8}$$

$$\phi_{-2}(z) = 720 + 203040q + 9417600q^2 + 223473600q^3 + 3566782080q^4 + \mathcal{O}(q^5), \tag{9}$$

$$\phi_0(z) = 518400q + 31104000q^2 + 870912000q^3 + 15697152000q^4 + \mathcal{O}(q^5). \tag{10}$$

Replacing z by it in (6), we see that

$$\phi_0(i/t) = \mathcal{O}(z^{-2}e^{2\pi t}), \quad \text{as } t \to \infty, \tag{11}$$

and by (10), we observe that

$$\phi_0(i/t) = \mathcal{O}(e^{-2\pi/t}), \quad \text{as } t \to 0. \tag{12}$$

It follows from these bounds that

$$a(r) := -4\sin\left(\frac{\pi r^2}{2}\right)^2 \int_0^{i\infty} \phi_0\left(-\frac{1}{z}\right) z^2 e^{i\pi r^2 z} dz. \tag{13}$$

converges absolutely for all $r > \sqrt{2}$, where we of course think of r as $\|x\|$, for $x \in \mathbb{R}^8$.

We first prove that

Proposition 5.1. *The function $r \mapsto a(r)$ continues analytically to a holomorphic function on a neighbourhood of $\mathbb{R}$. Moreover, its restriction to $\mathbb{R}$ is a Schwartz function and a radial eigenfunction of the Fourier transform on $\mathbb{R}^8$ with eigenvalue 1.*

Proof. First recall that

$$\sin\left(\frac{\pi\|x\|^2}{2}\right)^2 = \left(\frac{e^{i\pi\|x\|^2/2} - e^{-i\pi\|x\|^2/2}}{2i}\right)^2 = -\frac{1}{4}\left(e^{i\pi\|x\|^2} - 2 + e^{-i\pi\|x\|^2}\right).$$

Hence

$$\begin{aligned} a(r) = & \int_0^{i\infty} \phi_0\left(-\frac{1}{z}\right) z^2 e^{i\pi r^2(z+1)} dz - 2\int_0^{i\infty} \phi_0\left(-\frac{1}{z}\right) z^2 e^{i\pi r^2 z} dz && (14) \\ & + \int_0^{i\infty} \phi_0\left(-\frac{1}{z}\right) z^2 e^{i\pi r^2(z-1)} dz \\ = & \int_1^{1+i\infty} \phi_0\left(-\frac{1}{z-1}\right)(z-1)^2 e^{i\pi r^2 z} dz - 2\int_0^{i\infty} \phi_0\left(-\frac{1}{z}\right) z^2 e^{i\pi r^2 z} dz && (15) \\ & + \int_{-1}^{-1+i\infty} \phi_0\left(-\frac{1}{z+1}\right)(z+1)^2 e^{i\pi r^2 z} dz. \end{aligned}$$

Modifying the path of integration as in Figure 1, we obtain

$$\begin{aligned} a(r) = & \int_{-1}^{i} \phi_0\left(-\frac{1}{z+1}\right)(z+1)^2 e^{i\pi r^2 z} dz + \int_i^{i\infty} \phi_0\left(-\frac{1}{z+1}\right)(z+1)^2 e^{i\pi r^2 z} dz && (16) \\ & + \int_1^{i} \phi_0\left(-\frac{1}{z-1}\right)(z-1)^2 e^{i\pi r^2 z} dz + \int_i^{i\infty} \phi_0\left(-\frac{1}{z-1}\right)(z-1)^2 e^{i\pi r^2 z} dz \\ & - 2\int_0^{i} \phi_0\left(-\frac{1}{z}\right) z^2 e^{i\pi r^2 z} dz - 2\int_i^{i\infty} \phi_0\left(-\frac{1}{z}\right) z^2 e^{i\pi r^2 z} dz. \end{aligned}$$

Combining the intgerands on the paths from i to $i\infty$ and using the transformation law (6), we see that they simplify to $2\phi_0(z)e^{i\pi r^2 z}$. Hence,

$$a(r) = \int_{-1}^{i} \phi_0\left(-\frac{1}{z+1}\right)(z+1)^2 e^{i\pi r^2 z}dz + \int_{1}^{i} \phi_0\left(-\frac{1}{z-1}\right)(z-1)^2 e^{i\pi r^2 z}dz \qquad (17)$$
$$- 2\int_{0}^{i} \phi_0\left(-\frac{1}{z}\right) z^2 e^{i\pi r^2 z}dz + 2\int_{i}^{i\infty} \phi_0(z)e^{i\pi r^2 z}dz.$$

Now, using (12) and the rapid decay, quantitatively exponential, of $\phi_0(it)$ when $t \to \infty$, we see that the above expression is analytic on a neighbourhood of $\mathbb{R}$. We deduce that we obtained an analytic continuation of a to a neighbourhood of $\mathbb{R}$. The exponential decay of ϕ_0 as $\Im(z) \to \infty$ allows us to bound the terms in the previous equation and obtain that a and its derivative belong to a Schwartz space.

We now show that a is an eigenfunction of the Fourier transform on $\mathbb{R}^8$. The Fourier operator commutes with the integrals in (17) and we recall that

$$\mathcal{F}(e^{i\pi\|x\|^2 z})(y) = z^{-4} e^{i\pi\|y\|^2(\frac{-1}{z})}.$$

Therefore,

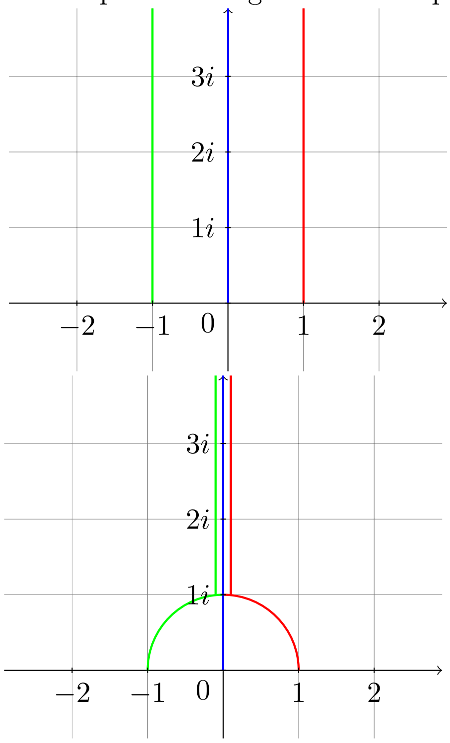
$$\begin{aligned}\mathcal{F}(a)(y) = &\int_{-1}^{i} \phi_0\left(\frac{-1}{z+1}\right)(z+1)^2 z^{-4} e^{i\pi\|y\|^2(\frac{-1}{z})}dz \\ &+ \int_{1}^{i} \phi_0\left(\frac{-1}{z-1}\right)(z-1)^2 z^{-4} e^{i\pi\|y\|^2(\frac{-1}{z})}dz \\ &- 2\int_{0}^{i} \phi_0\left(\frac{-1}{z}\right) z^2 z^{-4} e^{i\pi\|y\|^2(\frac{-1}{z})}dz + 2\int_{i}^{i\infty} \phi_0(z) z^{-4} e^{i\pi\|y\|^2(\frac{-1}{z})}dz.\end{aligned}$$

We now make the change of variable $\omega = \frac{-1}{z}$ and obtain

$$\begin{aligned}\mathcal{F}(a)(y) = &\int_{1}^{i} \phi_0\left(-1 - \frac{1}{\omega-1}\right)\left(\frac{-1}{\omega}+1\right)^2 \omega^2 e^{i\pi\|y\|^2\omega}d\omega \\ &+ \int_{-1}^{i} \phi_0\left(1 - \frac{1}{\omega+1}\right)\left(\frac{-1}{\omega}-1\right)^2 \omega^2 e^{i\pi\|y\|^2\omega}d\omega \\ &- 2\int_{i\infty}^{i} \phi_0(\omega) e^{i\pi\|y\|^2\omega}d\omega + 2\int_{i}^{0} \phi_0\left(\frac{-1}{\omega}\right)\omega^2 e^{i\pi\|y\|^2\omega}d\omega \\ =&a(y),\end{aligned}$$

where we used the $\mathbb{Z}$-periodicity of ϕ_0 to obtain the last equality. □

Figure 1: Deformation of the path of integration in the proof of Proposition 5.1



We finally check that a has the right vanishing behaviour at 0 and $\sqrt{2}$.

Proposition 5.2. *For all $r \geq 0$, we have*

$$a(r) = 4i \sin\left(\frac{\pi r^2}{2}\right)^2 \left(\frac{36}{\pi^3(r^2-2)} - \frac{8640}{\pi^3 r^4} + \frac{18144}{\pi^3 r^2} + \int_0^\infty \left(t^2 \phi_0\left(\frac{i}{t}\right) - \frac{36}{\pi^2}e^{2\pi t} + \frac{8640}{\pi}t - \frac{18144}{\pi^2}\right) e^{-\pi r^2 t} dt\right). \tag{18}$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

Proof. By (6) and the Fourier expansions of ϕ_{-2}, ϕ_{-4} and ϕ_0, we have that

$$\phi_0(i/t)t^2 = \phi_0(it)t^2 - \frac{12}{\pi}\phi_{-2}(it)t + \frac{36}{\pi^2}\phi_{-4}(it).$$

Hence

$$\phi_0(i/t)t^2 = \frac{18144}{\pi^2} + \frac{8640}{\pi}t + e^{2\pi t}\frac{36}{\pi^2} + \mathcal{O}(t^2 e^{-2\pi t}). \tag{19}$$

We note

$$p(t) := \frac{18144}{\pi^2} + \frac{8640}{\pi} t + e^{2\pi t} \frac{36}{\pi^2} \tag{20}$$

and

$$\tilde{p}(r) = \int_0^\infty p(t) e^{-\pi r^2 t} dt = \frac{36}{\pi^3 (r^2 - 2)} - \frac{8640}{\pi^3 r^4} + \frac{18144}{\pi^3 r^2}. \tag{21}$$

Notice that the expression in the statement is

$$d(r) := 4i \sin\left(\frac{\pi r^2}{2}\right)^2 \left[\tilde{p}(r) + \int_0^\infty \left(t^2 \phi_0(i/t) - p(t)\right) e^{\pi r^2 t} dt\right]. \tag{22}$$

Comparing zeroes and poles, we see that the expression outside of the integral is analytic in a neighbourhood of $\mathbb{R}$. Moreover, (19) shows that the integral is also analytic in a neighbourhood of $[0, \infty)$. We conclude that d extends analytically to a.

□

The vanishing behaviour of a can be directly read off the previous presentation, which also allows us to deduce the following:

Corollary 5.3. *We have*

$$a(0) = \frac{-i8640}{\pi}, \qquad a(\sqrt{2}) = 0, \qquad a^{'}(\sqrt{2}) = \frac{i72\sqrt{2}}{\pi}.$$

5.2 Construction of the -1 eigenfunction

Let us define the theta functions

$$\theta_{00}(z) = \sum_{n \in \mathbb{Z}} e^{i\pi n^2 z}, \tag{23}$$

$$\theta_{01}(z) = \sum_{n \in \mathbb{Z}} (-1)^n e^{i\pi n^2 z}, \tag{24}$$

$$\theta_{10}(z) = \sum_{n \in \mathbb{Z}} e^{i\pi \left(n + \frac{1}{2}\right)^2 z}. \tag{25}$$

These theta functions belong to the space $M_{1/2}(\Gamma(2))$, the space of modular forms of weight 1/2 with respect to the congruence subgroup

$$\Gamma(2) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod 2 \right\}.$$

To construct the -1-eigenfunction b, we first build the modular form

$$h := 128 \frac{\theta_{00}^4 + \theta_{01}^4}{\theta_{10}^8} \in M^!_{-2}(\Gamma_0(2)).$$

For $I = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, $S := \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, and $T := \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, all in $\mathrm{PSL}_2(\mathbb{Z})$, we continue by defining

$$\psi_I(z) := h(z) - h\,|_{-2}\,ST(z), \tag{26}$$

$$\psi_S(z) := \psi_I\,|_{-2}\,S(z) = \psi_I\left(\frac{-1}{z}\right)z^2, \tag{27}$$

$$\psi_T(z) := \psi_I\,|_{-2}\,T(z) = \psi_I(z+1). \tag{28}$$

Observe that

$$\psi_T\,|_{-2}\,S = -\psi_T, \tag{29}$$

$$\psi_I\,|_{-2}\,S = \psi_S \tag{30}$$

$$\psi_S\,|_{-2}\,S = \psi_I. \tag{31}$$

Additionally, we have that

Lemma 5.4. *The functions ψ_I, ψ_S and ψ_T verify the identity*

$$\psi_T + \psi_S = \psi_I. \tag{32}$$

Finally, define the radial function

$$b(r) := -4\sin\left(\frac{\pi r^2}{2}\right)^2 \int_0^{i\infty} \psi_I(z) e^{\pi i r^2 z} dz. \tag{33}$$

Viazovska then claims that

Proposition 5.5. *The function $r \mapsto b(r)$ extends analytically to holomorphic function on a neighbourhood of the real line and its restriction to $\mathbb{R}$ is a radial Schwartz function that verifies $\mathcal{F}(b) = -b$ for the 8-dimensional Fourier transform.*

Proof. The proof that b extends analytically to a holomorphic funciton on a neeighbourhood of the real line is essentially similar to the one for a, adjusting for the fact that we are here working with modular and not quasimodular forms. Similar estimates on ψ_I allow to justify the convergence of the integrals and the change in the path of integration is the same. The analytic continuation of b is given by

$$\begin{aligned} b(r) = &\int_{-1}^{i} \psi_T(z) e^{i\pi r^2 z} dz + \int_{1}^{i} \psi_T(z) e^{i\pi r^2 z} dz \\ &- 2\int_0^i \psi_I(z) e^{i\pi r^2 z} dz - 2\int_i^{i\infty} \psi_S(z) e^{i\pi r^2 z} dz \end{aligned} \tag{34}$$

on the larger domain, after making use of the identity given in the previous lemma.

Interchanging the Fourier transform and the path integral, doing the same change of variable as for a and using (29)–(31), Viazovska proves that $\mathcal{F}(b) - b$.

□

In addition, it can be proved that

Proposition 5.6. *For $r \geq 0$, we have*

$$b(r) = 4i \sin\left(\frac{\pi r^2}{2}\right)^2 \left[\frac{144}{\pi r^2} + \frac{1}{\pi(r^2-2)} + \int_0^\infty (\psi_I(it) - 144 - e^{2\pi t}) e^{-\pi r^2 t} dt\right]. \tag{35}$$

The integral converges absolutely for all $r \in \mathbb{R}_{\geq 0}$.

It follows that

Corollary 5.7. *We have*

$$b(0) = 0, \quad b(\sqrt{2}) = 0, \quad b'(\sqrt{2}) = 2\sqrt{2}\pi i. \tag{36}$$

6 Proof of Theorem 3.1

We have shown through the proof of different integral representation of a and b that they are eigenfunctions of the Fourier transform and that they obey the desired vanishing behaviour. We finally need to define g as a linear combination of these two functions and show that g verifies the inequalities of linear programming. Following Viazovska, we define

$$g(x) := \frac{i\pi}{8640} a(x) + \frac{i}{240\pi} b(x).$$

She then claims that the following holds:

Theorem 6.1. *The function g satisfies*

(1) $g(x) \leq 0$ for $\|x\| \geq \sqrt{2}$;

(2) $\mathcal{F}(g)(x) \geq 0$ for all $x \in \mathbb{R}^8$;

(3) $g(0) = \mathcal{F}(g)(0) = 1$.

Moreover, g and $\mathcal{F}(g)$ do not vanish for all vectors with $\|x\|^2 \notin 2\mathbb{Z}_{>0}$.

For $\|x\| > \sqrt{2}$ we have obtained

$$g(r) = \frac{\pi}{2160} \sin(\pi r^2/2)^2 \int_0^{i\infty} A(t) e^{-\pi r^2 t} dt,$$

where

$$A(t) = -t^2 \phi_0(i/t) - \frac{36}{\pi^2} \psi_I(it).$$

We want to show that $A(t) < 0$ for all $t \in (0, \infty)$ and we will see that this inequality follows from the spaces in which ϕ_0 and ψ_I where chosen, i.e. the space of quasimodular forms of depth 2 and the space of weakly holomorphic modular forms of weight -2 for $\Gamma_0(2)$. Recall that we had

$$\phi_0\left(\frac{-1}{z}\right) = \phi_0(z) - \frac{12i}{\pi}\frac{1}{z}\phi_{-2}(z) - \frac{36}{\pi^2}\frac{1}{z^2}\phi_{-4}(z);$$
$$\psi_S(z) = z^2\psi_I\left(\frac{-1}{z}\right).$$

Hence we obtain the presentations

$$A(t) = -t^2\phi_0(i/t) + \frac{36}{\pi^2}t^2\psi_S(i/t), \tag{37}$$
$$A(t) = -t^2\phi_0(it) + \frac{12}{\pi}t\phi_{-2}(it) - \frac{36}{\pi^2}\phi_{-4}(it) - \frac{36}{\pi^2}\psi_I(it). \tag{38}$$

Note that we can compute each of the expansions of the functions appearing in these expressions and hence obtain estimates on the asymptotic growth of A as t goes to 0 or to ∞, with any given degree of precision. More precisely, for any $n \geq 0$ we can write

$$A(t) = A_0^{(n)}(t) + \mathcal{O}(t^2 e^{-\pi n/t}), \quad t \to 0, \tag{39}$$
$$A(t) = A_\infty^{(n)}(t) + \mathcal{O}(t^2 e^{-\pi n t}), \quad t \to \infty, \tag{40}$$

for some functions $A_0^{(n)}$ and $A_\infty^{(n)}$.

The rest of the argument consists in bounding the truncation errors

$$R_0^{(n)} := |A(t) - A_0^{(n)}|, \tag{41}$$
$$R_\infty^{(n)} := |A(t) - A_\infty^{(n)}| \tag{42}$$

and comparing them to the size of the leading terms on the intervals $(0, 1]$ and $[1, \infty)$. Viazovska then deduces that

$$|R_0^{(6)}| \leq |A_0^{(6)}|, \quad \text{on } (0, 1], \tag{43}$$
$$|R_\infty^{(6)}| \leq |A_\infty^{(6)}|, \quad \text{on } [1, \infty). \tag{44}$$

Moreover,

$$A_0^{(6)} < 0, \quad \text{on } (0, 1], \tag{45}$$
$$A_\infty^{(6)} < 0, \quad \text{on } [1, \infty). \tag{46}$$

This clearly implies $A(t) < 0$ for $t \in (0, \infty)$.

Finally, Viazovska uses the same method on $\mathcal{F}(g)$, exploiting the same identities as above for ϕ_0 and ψ_I, to prove the second inequality of Theorem 3.1.

References

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