Sphere Packing - Hand out

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Contents

1 Statement of the problem

Consider $\mathbb{R}^n$ and let $r > 0$ be a strictly positive number. How can we arrange a collection $\mathcal{P}$ of non-overlapping spheres of radius r in $\mathbb{R}^n$ such that the volume between them is minimized? This collection $\mathcal{P}$ is called a *packing*, or a *sphere packing*. If $\mathcal{P}$ consists of a collection of non-overlapping copies of a subset $C \subset \mathbb{R}^n$, it is called a C-packing.

This naturally leads to the notion of the *density of a packing*.

Definition 1.1. *Let $A \subset \mathbb{R}^n$ be a bounded subset and let $\mathcal{P}$ be a packing of spheres. The* density $\Delta_{\mathcal{P},A}$ of $\mathcal{P}$ in A *is given by*

$$\Delta_{\mathcal{P},A} = \frac{\sum_{S \in \mathcal{P}} \mathrm{vol}(S \cap A)}{\mathrm{vol}(A)},$$

where, if $B \subset \mathbb{R}^n$, $\mathrm{vol}(B)$ simply denotes $\int_{\mathbb{R}^n} \chi_B$, the Lebesgue integral of the indicator function of B over the whole space.

We need to be subtler if we want to define densities of a packing over an unbounded subset of $\mathbb{R}^n$. In fact, one has to consider a limit.

Definition 1.2. *Let $\mathcal{P}$ be a packing, $x \in \mathbb{R}^n$ an arbitrary point and $r > 0$ a positive real number. We define the* density $\Delta_{r,x}$ of $\mathcal{P}$ around x *as*

$$\Delta_{r,x}(\mathcal{P}) = \frac{\mathrm{vol}(B_r(x) \cap \mathcal{P})}{\mathrm{vol}(B_r(x))},$$

where $B_r(x)$ is the ball of radius r around x.

Then this notion leads to the one of optimal packing density*, which is given by*

$$\Delta_n = \sup_{\mathcal{P}} \limsup_{r \to \infty} \Delta_{r,0}(\mathcal{P}).$$

These definitions of densities over unbounded subsets of $\mathbb{R}^n$ are only given for the sake of completeness. We won't encounter them in the following.

These notions of densities lead to natural questions, which constitute the sphere packing problem

- What is the optimal density?
- How can we compute it? Does there exist bounds that one cannot exceed?
- What is the optimal packing? How to find it?

1.1 Application

This problem has (at least) one technological application, i.e. communication over noisy channels. Consider the following simplified model. Suppose a radio antenna sends messages over some communication channel. One can represent the emitted signals by points in a bounded region of $\mathbb{R}^n$, say a centered ball around the origin of a large radius R, $B_R(0)$. The dimension n represents the number of measurements needed to describe the signals, for example, it can be a measurement of the amplitude of the emitted signals at n different frequencies. An emitted signal will automatically get some noise, i.e. a received signal r will not be the same as the emitted signal s. Is this possible to recognize s from r? In this simplified setting, one can imagine that there exists $\varepsilon > 0$, called the *error radius*, and an *error ball* $B_\varepsilon(s)$, such that r is recognized to be s if and only if $r \in B_\varepsilon(s)$.

The issue can occur in two different ways. On the one hand, if ε is too small and we don't have much different possibilities for *emitted signals* s, the collection of all the error balls $\{B_\varepsilon(s)\}$ will not cover a sufficient part of $B_R(0)$, and hence we won't be able to recognize a large amount of received messages. On the other hand, if ε is too large and a received message r appears in the non-empty intersection of two error balls $B_\varepsilon(s_1)$, $B_\varepsilon(s_2)$, we won't be able to recognize the emitted signal: "is r equal to s_1 or to s_2?"

In summary, we want the set of signals as large as possible, i.e. it has to cover the biggest fraction of $B_r(0)$ possible, but the error balls are not allowed to overlap. From this, we see where the sphere packing problem intervene.

2 Lattices

2.1 Definitions and properties

We want to get rid of the notion of unboundedness met in definition (1.2). This is one of the reason why we introduce lattices.

Definition 2.1. *A lattice $L \leq \mathbb{R}^n$ is a discrete subgroup of $\mathbb{R}^n$ of the form*

$$L = \Big\{\mathbb{Z}f_1 + \cdots + \mathbb{Z}f_m \,|\; f_j \in \mathbb{R}^n \text{ for } j \in \{1, \ldots, m\}\Big\},$$

where the f_j's are linearly independent. The collection $\{f_j\}_{j=1}^m$ is called a basis *of L.*

Lattices play an important role in many areas of the mathematics, as in Lie theory or differential geometry.

Example 2.2.

- *The lattice $\mathbb{Z}^n \leq \mathbb{R}^n$ is called* the standard lattice of $\mathbb{R}^n$.
- $\operatorname{span}_{\mathbb{Z}}\big\{(1,0), (\frac{1}{2}, \frac{\sqrt{3}}{2})\big\}$ *is a lattice in $\mathbb{R}^2$.*
- *The vector space* $\operatorname{Mat}_{m\times n}(\mathbb{R})$ *of matrices of size $m \times n$ has a lattice given by*

$$\bigoplus_{i=1,j=1}^{m,n} \mathbb{Z}E_{ij},$$

where E_{ij} is the matrix given by $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

A basis for a lattice is never unique. For example

$$\operatorname{span}_{\mathbb{Z}}\{(1,0),(0,1)\} = \operatorname{span}_{\mathbb{Z}}\{(1,0),(1,1)\},$$

are two bases for the standard lattice of $\mathbb{R}^2$.

Definition 2.3. *Let $L \leq \mathbb{R}^n$ be a lattice with basis $\{f_1, \ldots, f_m\} \subset \mathbb{R}^n$. The* generator matrix $M \in \operatorname{Mat}_{m\times n}(\mathbb{R})$ *for L is defined as the matrix*

$$M = \begin{pmatrix} f_1^T \\ \hline \vdots \\ \hline f_m^T \end{pmatrix}.$$

By definition, it follows that $L = \{vM\}_{v \in \mathbb{Z}^n}$.

Definition 2.4. *Let $L \leq \mathbb{R}^n$ be a lattice with a generator matrix M. The matrix $A := MM^T$ is called the* Gram matrix *of L.*

Definition 2.5. *Let L be a lattice with generator matrix M and Gram matrix A, define the determinant $\det(L)$ of a lattice L to be*

$$\det(L) = \det(A).$$

Definition 2.6. *Let $L \leq \mathbb{R}^n$ be a lattice with basis $\{f_1, \ldots, f_m\} \subset \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ be an arbitrary point. The* orbit $\mathrm{Orb}(x)$ *of x is defined as the set*

$$\mathrm{Orb}(x) = \big\{x + \sum_{j=1}^{m} a_j f_j |\ a_j \in \mathbb{Z}\big\}.$$

Definition 2.7. *Let $L \leq \mathbb{R}^n$ be a lattice. A* fundamental region *of L is a subset $F \subset \mathbb{R}^n$ such that*

- *any two different elements $a, b \in F$ are in two distinct orbits,*
- *the union of the orbits of the elements of F is $\mathbb{R}^n$.*

Example 2.8. *Let $L \leq \mathbb{R}^n$ be a lattice and let $\{f_j\}_{j=1}^n \subset \mathbb{R}^n$ be a basis. The set*

$$F = \big\{ \sum_{j=1}^{n} a_j f_j |\ 0 \leq a_j < 1\big\}.$$

is called the standard fundamental region *of L. In fact, if $v \in L$, $v + F$ is again a fundamental region.*

Fundamental regions are interesting because they contain only one lattice point.

Let us focus now on the case where $L \leq \mathbb{R}^n$ is a lattice of the form

$$L = \bigoplus_{i=1}^{n} \mathbb{Z} f_i.$$

Proposition 2.9. *Let $L \leq \mathbb{R}^n$ be an n-dimensional lattice. Then $\det(L)$ does not depend on the lattice basis we choose to compute it.*

Proof. Let $\mathcal{B} = \{f_i\}_{i=1}^n$ and $\mathcal{B}' = \{g_j\}_{j=1}^n$ be two bases for L and let M and M' be the two generator matrices induced by $\mathcal{B}$ and by $\mathcal{B}'$ respectively.

As $\mathcal{B}$ and $\mathcal{B}'$ are two bases for L, there exists a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ with determinant ± 1 such that for every $i \in \{1, \ldots, n\}$, $Af_i = g_i$. Then, the following holds

$$\begin{aligned}
\det(M') &= \det(g_1, \ldots, g_n) \\
&= \det(Af_1, \ldots, Af_n) \\
&= \det(A)\det(f_1, \ldots, f_n) \\
&= \pm\det(f_1, \ldots, f_n) \\
&= \pm\det(M).
\end{aligned}$$

Hence $\det(M)^2 = \det(M')^2$ and consequently $\det(L)$ does not depend on the choice of a basis. $\square$

Definition 2.10. *Let $L \leq \mathbb{R}^n$ be a n-dimensional lattice. Let $C \subset \mathbb{R}^n$ be an arbitrary subset. A* lattice packing *of C is given by the set*

$$\mathcal{P} = \{C + v\}_{v \in L},$$

where we assume that for any $v \neq w \in L$, then $C + v \cap C + w = \emptyset$. The density Δ of a lattice packing $\mathcal{P}$ of C *is given by*

$$\Delta = \frac{\text{vol}(C)}{\det(L)^{1/2}}.$$

2.2 Dynkin diagrams

Some lattices have interesting properties, and we want to encode these informations in a so called *Dynkin diagram*. Such a diagram allows us, among others, to find a basis of a lattice.

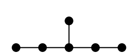
Definition 2.11. *Let $L \leq \mathbb{R}^n$ be a lattice with basis $\mathcal{B} = \{f_1, \dots, f_m\}$. Suppose $\mathcal{B}$ has the particularity that for every i,j $\in \{1, \dots, m\}$, there exists $p_{ij} \in \mathbb{N}$ such that*

$$\frac{\langle f_i, f_j \rangle}{||f_i||||f_j||} = -\cos(\frac{\pi}{p_{ij}}).$$

A Dynkin diagram *of L is an undirected graph G with vertices $\{a_1, \dots, a_m\}$, where a_i corresponds to f_i for every $i \in \{1, \dots, m\}$ and labeled edges (a_i, a_j), whenever $p_{ij} \geq 3$. The label on every edge is equal to the corresponding p_{ij}.*

Remark 2.12. *More commonly, if $p_{ij} = 3$, one does not write 3, if $p_{ij} = 4$, one does not write 4, but one doubles the edge and if $p_{ij} = 6$, one does not write 6 but one triples the edge.*

Example 2.13. *Here are two Dynkin diagrams. The first one describes the lattice E_6 and the second one the lattice F_4.*



Example 2.14. *Let $L \leq \mathbb{R}^4$ be the 3-dimensional lattice spanned by the vectors $\{f_1 = e_2 - e_1, f_2 = e_3 - e_2, f_3 = e_4 - e_3\}$. Then*

$$\frac{\langle f_1, f_2 \rangle}{||f_1||||f_2||} = -\frac{1}{2} = -\cos(\frac{\pi}{3}).$$

Hence $p_{12} = 3$. Similarly

$$\frac{\langle f_1, f_3 \rangle}{||f_1||||f_3||} = 0 = -\cos(\frac{\pi}{2}),$$

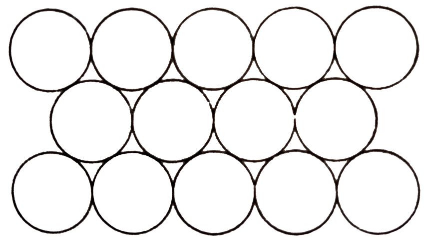


Figure 1: The 2-dimensional hexagonal packing.

$p_{13} = 2$ *and*

$$\frac{\langle f_2, f_3 \rangle}{||f_2|| ||f_3||} = -\frac{1}{2} = -\cos(\frac{\pi}{3}),$$

$p_{23} = 3$.

Consequently, the Dynkin diagram of L is given by

3 Summary of results in small dimensions

These results are taken from [3]. The best sphere packings are only known for $n = 1, 2, 3, 8$ and 24, but for $n = 4, \dots 7$, it is expected that the best sphere packing is a lattice packing.

3.1 1-dimensional packing

This is trivial. One can take 1-dimensional spheres, i.e. intervals on length 1 and center them at integer points. The optimal density is then 1. The lattice given by this packing has Dynkin diagram A_1

3.2 2-dimensional packing

The highest density that can be achieved is $\frac{\pi}{2\sqrt{3}} = 0.9068...$ The optimal packing is given by the hexagonal packing as it can be seen in figure 1. A short proof of the optimality of this packing can be found in [2]. The lattice given by this packing has Dynkin diagram A_2

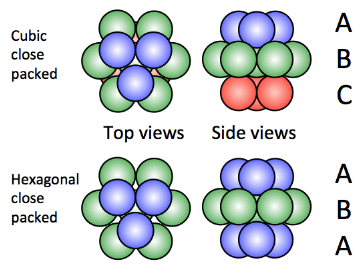


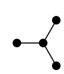
Figure 2: The fcc and hcp sphere packing.

3.3 3-dimensional packing

The answer for a general packing is known. There are multiple different packings that achieves the optimal density of $\frac{\pi}{3\sqrt{2}} = 0.7404....$ One of them is given by the face centered cubic lattice (fcc). The hexagonal closed packing (hcp) is another packing that achieves this density. Both patterns of those packings can be seen in figure 2. There also exists non-lattice sphere packing that are as dense as the face centered cubic packing. The lattice of the fcc packing has Dynkin Diagram A_3

3.4 4-dimensional packing

The densest lattice packing known until this day is given by the lattice with Dynkin diagram D_4



The centers of the spheres are given by the set C,

$$C = \Big\{(a_1, a_2, a_3, a_4) \in \mathbb{Z}^4 | \sum_{i=1}^{4} a_i \text{ is an even number}\Big\} \subset \mathbb{R}^4.$$

The sphere at the origin touches 24 other spheres and if one set the radius to be $\frac{\sqrt{2}}{2}$, they are touching each other, but do not overlap. One can then compute the density.

This lattice has generator matrix

$$M = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

and hence $\det(D_4) = \det(M)^2 = 4$. Note that rows of M do not correspond to the given Dynkin diagram.

The volume V of a sphere of radius 1 in $\mathbb{R}^4$ is given by

$$V = \frac{\pi^2}{2}$$

and the volume of a sphere W of an arbitrary radius r in $\mathbb{R}^4$ is given by

$$W = Vr^4.$$

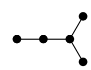
Consequently, the density Δ of the lattice packing is given by

$$\Delta = \frac{W}{\det(L)^{1/2}} = \frac{\pi^2}{16} = 0.6169...$$

Moreover, it is proven that this is the densest lattice packing in $\mathbb{R}^4$.

3.5 5-dimensional packing

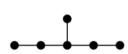
In $\mathbb{R}^5$, the best packing is expected to be a lattice packing with a lattice that has Dynkin diagram D_5



The density of this packing is $\frac{\pi^2}{15\sqrt{2}} = 0.4653...$

3.6 6-dimensional packing

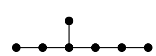
In $\mathbb{R}^6$, the best packing is expected to be a lattice packing with a lattice that has Dynkin diagram E_6



The density of this packing is 0.3730....

3.7 7-dimensional packing

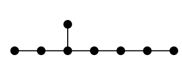
In $\mathbb{R}^7$, the best packing is expected to be a lattice packing with a lattice that has Dynkin diagram E_7



The density of this packing is 0.2953....

3.8 8-dimensional packing

It has been proven by Viazovska in [4] that no sphere packing in $\mathbb{R}^8$ has a density greater than the one induced by the lattice E_8. This lattice has Dynkin diagram E_8



and the density of this packing is 0.2537....

To summarize, the dimension, the lattice and the corresponding density are given here:

n	1	2	3	4	5	6	7	8
Lattice	A_1	A_2	A_3	D_4	D_5	E_6	E_7	E_8
Density	1	0.9069	0.7405	0.6169	0.4653	0.3730	0.2953	0.2537

4 Non constructive lower bounds for optimal density

As it can be guessed in section 3, the optimal density of high dimensional packings seems to drop when the dimension increases. The only thing we know about the optimal density is lower and upper bounds. Moreover, the ratio between these two bounds grows *exponentially* as $n \to \infty$. The reason to this unexpected phenomena comes from the fact that the volume of an hypercube in $\mathbb{R}^n$ of a certain size length a is given by a^n. Hence it is exponential in n. Then if one takes a packing in $\mathbb{R}^n$ and move the centers of the spheres 1% further apart, the density will be lowered by a factor of 1.01^n, which is insignificant in low dimensions, but enormous as n grows as it is explained in [1].

4.1 A first lower bound

Here is our first result about a lower bound on the optimality, but first,

Definition 4.1. *Let $\mathcal{P}$ be a sphere packing in $\mathbb{R}^n$. This packing is said to be* saturated *if no sphere of the same radius as the one in $\mathcal{P}$ can be added to $\mathcal{P}$ without overlapping a sphere of the packing.*

Theorem 4.2. *Let $\mathcal{P}$ be a saturated sphere packing in $\mathbb{R}^n$. Then $\mathcal{P}$ has density at least 2^{-n}.*

Proof. Without loss of generality, suppose the radius of the spheres in $\mathcal{P}$ is equal to 1. Let $x \in \mathbb{R}^n$ be an arbitrary point. Then there exists a sphere $S \in \mathcal{P}$ with radius c such that $||x - c|| < 2$. Indeed, if not, one can add a sphere $B_1(x)$ to $\mathcal{P}$. As the packing is assumed to be saturated, this is impossible.

This means that when one doubles the radius of the spheres in $\mathcal{P}$, one obtains a cover of $\mathbb{R}^n$. But doubling the radius implies that the volume of spheres is multiplied by a factor 2^n. Hence, a 2^{-n} fraction of $\mathbb{R}^n$ must be covered originally by $\mathcal{P}$. Consequently the density of $\mathcal{P}$ is at least $\mathbb{R}^n$. □

4.2 The Minkowski-Hlawka theorem

Definition 4.3. *Let $C \subset \mathbb{R}^n$. Then C is a* centrally symmetric convex body *if*

- *it contains the origin,*
- *C is convex,*
- *for any $x \in C$, $-x \in C$.*

If C is a centrally symmetric convex body, then there exists a lattice packing of C that achieves the optimal lattice density, denoted $\Delta_{\text{opt}}(C)$. We go forward and give a better lower bound for saturated C-packing. This is the content of the Minkowski-Hlawka theorem. A proof can be found in [5].

Theorem 4.4 (Minkowski-Hlawka). *Let C be an n-dimensional centrally symmetric convex body. Then*

$$\Delta_{\text{opt}}(C) \geq \frac{\zeta(n)}{2^{n-1}},$$

where $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}$ is the Riemann zeta function.

Note that the lower bound given by theorem 4.4 is almost twice the one given by theorem 4.2, as $\lim_{n\to\infty} \zeta(n) = 1$.

5 References

[1] Henry Cohn. *Packing, coding and ground states.* URL: https://arxiv.org/pdf/1603.05202.pdf.

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[3] N. J. A. Sloane J. H. Conway. *Sphere Packings, Lattices and Groups.* 1999. Chap. 1. ISBN: 978-1-4419-3134-4.

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[5] C. Zong. *Sphere Packings.* 1999. Chap. 3. ISBN: 0-387-98794-0.