Sphere Packings, Lattices and Codes: Talk 10

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Contents

Our goal today is to define the so called *theta function*. And discuss some of its properties and applications.

1 Recap and definitions

We first recall, what lattices are. We encountered them throughout the seminar. Consider an $\mathbb{R}$-vector space V of dimension n.

Definition 1.1 (Lattice)**.** A **lattice** Λ is a subgroup of V with the following condition:

There exists an $\mathbb{R}$-basis $(a_1, \ldots, a_n)$ of V with which we can write our lattice as

$$\Lambda = \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_n,$$

together with a symmetric positive definite bilinear form $\langle \cdot, \cdot \rangle$.

In other words, $\{a_i\}_i$ are a $\mathbb{Z}$-basis of Λ, so we can write elements from the lattice as $x := (x_1, \ldots, x_n) \in \Lambda$, where $x_i \in \mathbb{Z}$.

Definition 1.2 (integral, even, odd lattices)**.** A lattice is called **integral**, if $\langle x, x \rangle \in \mathbb{Z} \quad \forall x \in \Lambda$.

We call an integral lattice **even**, if in addition $\langle x, x \rangle$ is even. An integral lattice that is not even is called **odd**.

Example. As an easy example, take the lattice Λ in $\mathbb{R}^2$ spanned by the basis vectors $\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ with the scalar product as the associated bilinear form: Take now two arbitrary elements $\begin{pmatrix} 2a \\ 2b \end{pmatrix}, \begin{pmatrix} 2c \\ 2d \end{pmatrix} \in \Lambda \quad (a, b, c, d \in \mathbb{Z})$, then the scalar product of those two elements is always even (and in particular integral). So Λ is an even lattice.

Now recall from linear algebra the dual vector space V'. In the same way, we can define a dual lattice Λ' to Λ.

Definition 1.3 (Dual lattice)**.** Let Λ be a lattice. Its dual lattice is defined by

$$\Lambda' := \{y \in V' | \langle x, y \rangle \in \mathbb{Z} \quad \forall x \in \Lambda\}$$

In the following, we will consider a special kind of lattices, namely the ones that coincide with their dual lattices.

Definition 1.4 (Unimodular lattice)**.** A lattice is called unimodular, if it coincides with its dual lattice. Equivalently, this means that the corresponding basis matrix has determinant ± 1.

Example.

- The Λ_8 lattice is an unimodular lattice.

$$\Lambda_8 = \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8 : \sum_i x_i = 0 \mod 2\}$$

  In words, the lattice of points whose coordinates are all integers or are all half integers (no mix allowed) and the sum of the coordinates is even. To show this, consider a basis of Λ_8

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

  The determinant of this matrix is 1.

- The Leech Lattice (cf. Talk 5) is unimodular. It is also even.

Definition 1.5 (Quadratic forms)**.** Let M be an R-module for a commutative ring R. A quadratic form $Q : M \to R$ satisfies

$$Q(ax) = a^2 Q(x) \quad \forall a \in R, x \in M$$

and

$$b : M \times M \to R, (x, y) \mapsto Q(x + y) - Q(x) - Q(y)$$

is bilinear.

For us, this just means $Q(x_1, \ldots, x_m)$ is a polynomial $\mathbb{Z}^m \to \mathbb{Z}$ with integer coefficients where all terms are of degree 2.

Alternatively, we can write a quadratic form as

$$Q(x) = x^T A x,$$

for $A \in \mathbb{Z}^{m \times m}$ a symmetric matrix and $x \in \mathbb{Z}^m$.

Example.

$$\begin{aligned} Q :&\mathbb{Z} \to \mathbb{Z} \\ &x \mapsto x^2 \end{aligned}$$

is called the *unary quadratic form.*

I believe (or hope) that most of us still remember the Fourier transform from the 3rd semester course *Methods of Mathematical Physics I* (*Mmp1*). If not, here is a reminder.

Definition 1.6 (Fourier transform)**.** let f a function with $\int_{\mathbb{R}} |f| dx < \infty$. Its Fourier transform is defined by

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2i\pi kx} dx$$

Example.

- Let $f(x) = e^{-\pi t x^2}$, then its Fourier transform is $\hat{f}(k) = \frac{e^{-\pi k^2/t}}{\sqrt{t}}$.
- $f(x) = e^{-\pi x^2}$ is its own Fourier transform, i.e. $\hat{f}(k) = e^{-\pi k^2}$.

With that I also want to remind you of the Poisson summation formula.

Lemma 1.7.

$$\sum_{n\in\mathbb{Z}} f(x) = \sum_{k\in\mathbb{Z}} \hat{f}(k)$$

Now for the last recap, remember last weeks's talk about modular forms.

Definition 1.8 (Modular group)**.** We have the special linear group $SL_2(\mathbb{Z})$. The modular group is defined as the projective linear group

$$PSL_2(\mathbb{Z}) := SL_2(\mathbb{Z})/\{\pm 1\},$$

usually denoted by Γ.

Definition 1.9 (Modular forms)**.** Let $k \in \mathbb{Z}$. A function $f : \mathbb{H} \to \mathbb{C}$ is called modular form of weight k if it has the following properties:

i) f is holomorphic on $\mathbb{H}$,

ii) $f(M.z) = (cz+d)^k f(z)$, where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $M.z := \frac{az+b}{cz+d}$,

iii) f has a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n,$$

where $q = e^{2\pi i z}$.

ii) is also called the modular transformation property.

Example.

- Eisenstein series
- Modular j-invariant
- Lattice invariant Δ

1.1 The q-expansion principle

From here on, we will denote by $q := e^{2\pi i z}$ for some $z \in \mathbb{H}$. There is a very convenient technique to decide, whether two modular forms (of the same wight) are the same.

Let f and g be modular forms both of weight k and consider their Fourier expansion:

$$f(z) = \sum_n a_n q^n$$
$$g(z) = \sum_n b_n q^n.$$

It suffices to check for equality $a_n = b_n$ of coefficients only for $n = 0, 1, \ldots, C(k) - 1$, where $C(k)$ depends only on the weight k.

In our case, we can take $C(k)$ to be the dimension of space of modular forms of weight k.

2 Our first Theta Series

We define now a first, very simple theta series. For these we generally use the *nome* $q := e^{2\pi i z}$ Consider the unary quadratic form

$$\begin{aligned} Q :& \mathbb{Z} \to \mathbb{Z} \\ & x \mapsto x^2. \end{aligned}$$

We can associate a theta function to this form which looks like this.

$$\theta(z) = \sum_{x \in \mathbb{Z}} q^{x^2} = 1 + 2q + 2q^4 + 2q^9 + \ldots$$

First of all, we want to show that this is a modular form. (Reminder: Def. 1.9)

Since we defined our θ-function in the form of a Fourier-expansion, we already satisfy *iii)* and we can easily see that the series is holomorphic on $\mathbb{H}$ (which means *i)*).

That means, we have to show the functional equation

$$f(M.z) = (cz + d)^k f(z),$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $M.z := \frac{az+b}{cz+d}$. The trick here is, we only need to check this for generators of the group Γ.

You might remember the following two transformations from last week:

$$\begin{aligned} z &\mapsto z + 1 \\ z &\mapsto -1/z \end{aligned}$$

Generally, if we show the functional equation with these two (or in fact with any two generators), then the equation holds for the whole group.

I will use two different transformations:

$$\begin{aligned} &z \mapsto z + 1, \text{ as before, and} \\ &z \mapsto -1/(4z) \end{aligned}$$

Or in other words, $\theta(z)$ satisfies the equations:

$$\theta(z+1) = \theta(z), \qquad \theta(-\frac{1}{4z}) = \sqrt{\frac{2z}{i}}\theta(z) \qquad (z \in \mathbb{H}).$$

which come from the two matrices

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S_N := \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}.$$

These two generate the so called "congruence subgroup"

$$\Gamma_0(4) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \mod 4 \right\}$$

Proof of the functional equations. The first equation can be seen immediately. Just plug in the definitions:

$$\begin{aligned} \theta(T.z) = \theta(z+1) &= \sum_{n \in \mathbb{Z}} e^{2\pi i (z+1) n^2} \\ &= \sum_{n \in \mathbb{Z}} e^{2\pi i z n^2} \cdot \underbrace{e^{2\pi i n^2}}_{=1} \\ &= \theta(z). \end{aligned}$$

For the second equation, we will need the previously mentioned Poisson transformation formula. Remember our first example of Fourier transforms:

$$f(x) = e^{-\pi t x^2} \text{ with Fourier transform } \hat{f}(k) = \frac{e^{-\pi k^2/t}}{\sqrt{t}}.$$

Plugging this into the Poisson formula gives

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} \quad (t > 0).$$

This proves our second equation for $z = it/2$ (which lies on the positive imaginary axis)

$$\begin{aligned} \theta(z)|_{z=it/2} = \theta(it/2) = \sum_{n \in \mathbb{Z}} q^{n^2} &= \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} \\ = \frac{1}{\sqrt{t}} \sum_{n \in \mathbb{Z}} e^{-\pi n^2/t} &= \left(\sqrt{\frac{2z}{i}} \sum e^{2\pi i (-\frac{1}{4z})} \right)_{z=it/2} \end{aligned}$$

and the general case follows from analytic continuation. $\square$

With this we proved, that $\theta(z)$ is a modular form of weight $1/2$. But so far, we've only defined modular forms of integer weight, so for us, we can just interpret this as "$\theta(z)^2$ is a modular form of weight 1".

2.1 Application of the Unary Theta Series

Now this definition is great and all, but what do we do with this? Consider the following question:

Can we write certain numbers as a sum of 4 squares?

Which is actually the well known Theorem of Lagrange

Theorem 2.1. *Every positive integer is a sum of four squares.*

We will prove this theorem now, using modular forms and the unary theta series.

Take a look at $\theta(z)^4$, a modular form of weight 2. If we write it out in a bit more detail, it looks like this:

$$(\sum_{a\in\mathbb{Z}} q^{a^2})(\sum_{b\in\mathbb{Z}} q^{b^2})(\sum_{c\in\mathbb{Z}} q^{c^2})(\sum_{d\in\mathbb{Z}} q^{d^2})$$

If we now calculate this product of sum, we arrive at the series

$$\theta(z)^4 = \sum_{n\in\mathbb{Z}} r_4(n)q^n,$$

where $r_4(n) := |\{(a,b,c,d) \in \mathbb{Z}^4 | a^2+b^2+c^2+d^2 = n\}|$, or in other words, the number of representations of n as a sum of 4 squares.

From last week we know that the spaces of modular forms M_k have a surprisingly **low** dimension, in particular the space of modular forms of weight $k = 2$ is only two-dimensional. Let's try and find a basis of it.

Recall the Eisenstein series from last week:

$$G_k(z) := \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k} \qquad (k>2, z\in\mathbb{H})$$

$$\mathbb{G}_k(z) := \frac{(k-1)!}{(2\pi i)^k} G_k(z),$$

where $\mathbb{G}_k(z)$ is a helpful normalization of the Eisenstein series.

In our case, it is more convenient, to use the Fourier expansion.

$$\mathbb{G}_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

where B_k is the k-th Bernoulli number and $\sigma_{k-1}(n) := \sum_{\substack{d|n \\ d>0}} d^{k-1}$.

As we know, the Eisenstein series are modular forms for $k>2$, but sadly we are working with weight exactly $k=2$. To fix this, let's just define the Eisenstein series with weight 2 and modify them to fit with modular forms.

We take the definition (q-expansion) of $\mathbb{G}_k$ for $k=2$ and see what happens:

$$\mathbb{G}_2(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n = -\frac{1}{24} + q + 3q^2 + 4q^3 + 7q^4 + 6q^5 + \ldots$$

We can easily check that this does **not** satisfy the modular transformation property anymore by plugging in some examples (I'm skipping this here, though). The problem is a lingering error term, when we try to do so. We want to fix this of course, so lets try to get rid of the error term. Define

$$\mathbb{G}_2^*(z) := \mathbb{G}_2(z) + \frac{1}{8\pi y},$$

where $y = \mathfrak{Im}(z)$, which now **does** satisfy the modularity property (for $\Gamma_0(4)$) but is, instead, **not holomorphic** anymore.

It's easy to see, that $\mathbb{G}_2^*(2z)$, as well as $\mathbb{G}_2^*(4z)$ transform in the same way (but still are not holomorphic!).

As a last step, look at the following two linear combinations

$$\mathbb{G}_2^*(z) - 2\mathbb{G}_2^*(2z) = \mathbb{G}_2(z) - 2\mathbb{G}_2(2z) \quad \text{and}$$
$$\mathbb{G}_2^*(2z) - 2\mathbb{G}_2^*(4z) = \mathbb{G}_2(2z) - 2\mathbb{G}_2(4z).$$

They are both holomorphic **and** have the modularity property **and** are linearly independent. Together, they give the desired basis for $M_2(\Gamma_0(4))$.

By comparing only the first two coefficients in $\theta(z)^4 = 1 + 8q + \dots$, we now find that $\theta(z)^4$ equals

$$8(\mathbb{G}_2(z) - 2\mathbb{G}_2(2z)) + 16(\mathbb{G}_2(2z) - 2\mathbb{G}_2(4z)),$$

or expressed in words:

Proposition 2.2. *Let n be a positive integer. Then the number of representations of n as a sum of four squares is 8 times the sum of the positive divisors of n which are not multiples of 4.*

And in particular, the coefficient of q^n for any n in above modular form is non-zero, which proves Lagrange's Theorem.

3 Theta Series through Quadratic Forms

now, we want to further generalize our theta series by choosing arbitrary quadratic forms $Q : \mathbb{Z}^m \to \mathbb{Z}$ instead of the unary one. Here as well, $q = e^{2\pi i z}$ for $z \in \mathbb{H}$. Define:

$$\theta_Q(z) := \sum_{x_1,\dots,x_m \in \mathbb{Z}} q^{Q(x_1,\dots,x_m)},$$

which we can also write as

$$\theta_Q(z) = \sum_{n \geq 0} R_Q(n) q^n.$$

One can show that this is a modular form of weight $m/2$. In fact, the following theorem holds:

Theorem 3.1. *Let $Q : \mathbb{Z}^{2k} \to \mathbb{Z}$ be a positive definite integer-valued form in $2k$ variables of level N and discriminant Δ. Then θ_Q is a modular form on $\Gamma_0(N)$ of weight k and character χ_Δ, i.e., we have*

$$\theta_Q\left(\frac{az+b}{cz+d}\right) = \chi_\Delta(a)(cz+d)^k \theta_Q(z)$$

for all $z \in \mathbb{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

Where we have certain new terms, that might be confusing, but are not of further essence in this talk.

The proof of this theorem also relies on the Poisson summation formula which gives us the identity

$$\theta_G(-1/Nz) = N^{k/2}(z/i)^k \theta_{Q^*}(z),$$

where $Q^*(x)$ is a slightly different quadratic form. To find the precise modular behaviour uses a lot of work, so we won't be doing this here.

Another way to prove the above theorem is to break the higher rank case down to the one-variable case, by utilizing the fact, that quadratic forms are diagonalizable over $\mathbb{Q}$. The sum for $\theta_Q(z)$ can be broken up into many sub-sums on which the quadratic form can be written as a linear combination of m squares.

I will not go into further details, instead I will show you another approach to theta functions.

4 Theta Series through Lattices

Let Λ be an **even unimodular** lattice in a vector space V of dimension n. Let $(\cdot, \cdot)$ be its bilinear form. For an integer m define $r_\Lambda(m)$ as the number of elements of Λ such that $(x, x) = 2m$. Here yet again, $q = e^{2\pi i z}$ for $z \in \mathbb{H}$.

With this, let's define a theta function

$$\begin{aligned} \theta_\Lambda(z) &:= \sum_{m=0}^{\infty} r_\Lambda(m) q^m \\ &= \sum_{x \in \Lambda} q^{(x,x)/2} = \sum_{x \in \Lambda} e^{\pi i z (x,x)}. \end{aligned}$$

We can show in a very similar way (i.e. through the Poisson formula) that this is a modular form of weight $n/2$. The modular transformation property for $z \mapsto -1/z$ looks like this:

$$\theta_\Lambda(-1/z) = (iz)^{n/2} \theta_\Lambda(z).$$

An interesting fact about this is the condition that even unimodular lattices put on the vector space V they come from.

Proposition 4.1. *The dimension n of V is divisible by 8.*

Quickly recall the two generators for Γ:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and calculate

$$(ST)^3 = I.$$

Proof. Suppose n is not divisible by 8. By replacing Λ by $\Lambda \oplus \Lambda$ or $\Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda$, we may supose that $n \equiv 4 \mod 8$. We can now rewrite the above modularity property by

$$\theta_\Lambda(-1/z) = (-1)^{n/4} z^{n/2} \theta_\Lambda(z) = -z^{n/2} \theta_\Lambda(z).$$

Define the differential $\omega(z) = \theta_\Lambda(z) dz^{n/4}$ and see that it transforms into $-\omega$ by $S : z \mapsto -1/z$, while it stays invariant under $T : z \mapsto z + 1$.

But then ω transforms into $-\omega$ under ST, which contradicts $(ST)^3 = I$. $\square$

One thing we can immediately see from the theta function over lattices is the coefficient of q^0. Since the bilinear form is positive definite, there is only one element in $x \in \Lambda$ that satisfies $(x, x) = 2 \cdot 0$, namely $x = 0$. This means, $r_\Lambda(0) = 1$, thus every θ_Λ starts with

$$\theta_\Lambda(z) = 1 + \dots$$

In the following are two corollaries that follow from the proof of the modular transformation property of θ_Λ, which I won't actually be proving rigorously here.

Corollary 4.2. *There exists a cusp form f_Λ of weight $n/2$ such that*

$$\theta_\Lambda = E_{n/2} + f_\Lambda$$

This follows from the fact that $\theta_\Lambda(\infty) = 1$ and $f_\Lambda(\infty) = 0$, as it is a cusp form.

Corollary 4.3. *We have $r_\Lambda(m) = \frac{n}{B_{n/2}} \sigma_{n/2-1}(m) + \mathcal{O}(m^{n/4})$*

4.1 Lattices vs Quadratic forms

Now, one could ask the question. What are the differences between the theta functions defined through quadratic forms an the ones defined through lattices. And the answer is, essentially none.

In fact, one can go between lattices and quadratic forms through an easy procedure and I will show this really wuickly before we go to the last topic.

We defined lattices Λ with a bilinear form $b : V \times V \to R$. By finding the Gram matrix through the bilinear form and the basis matrix of Λ, we find a symmetric matrix we can associate to a quadratic form, and vice versa.

5 Applications to the Leech Lattice and the Λ_8-Lattice

5.1 Leech Lattice

We are now looking for the amount of points in the Leech lattice of length exactly $\sqrt{2m}$ ($m \in \mathbb{Z}$).

Recall the definition of the Leech lattice from talk 5:

Definition 5.1 (Leech Lattice Λ_{24})**.**

$$\Lambda_{24} := \frac{1}{\sqrt{2}}\left(A \cup \left(\frac{1}{2} \cdot 1 + N\right)\right)$$

where $1 := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ and A and N are defined as follows:

Remember the construction of the lattice $\Gamma_C := \frac{1}{\sqrt{2}}\rho^{-1}(C)$ for a linear code C with ρ the canonical projection $\mathbb{Z}^n \to \mathbb{F}_2^n$. We use this on the (extended binary) Golay code G (where $n = 24$, i.e. we look at the lattice Γ_G. We then define a map $\alpha : \Gamma_G \to \mathbb{F}_2, \quad x \mapsto \frac{1}{2}\sum x_i \mod 2$.

With this we define $A := \alpha^{-1}(0)$ and $N := \alpha^{-1}(1)$.

This lattice is (up to isomorphism) the unique even unimodular lattice of rank 24. Unimodularity follows from the unimodularity from Γ_G which was also already mentioned a few weeks ago.

Now what do we do with this? First of all, lets construct the theta series of this lattice:

$$\theta_{\Lambda_{24}}(z) = \sum_{m \geq 0} r_{\Lambda_{24}}(m) q^m$$

The coefficient $r_{\Lambda_{24}}(m)$ now tells us (by definition even) exactly how many lattice points in Λ_{24} of length $\sqrt{2m}$ there are.

As we established, $\theta_{\Lambda_{24}}(z)$ is a modular form of weight 12. The space of modular forms of weight 12 has dimension 2, and a possible basis are the following two functions.

$$\begin{aligned} E_{12} &= 1 + \frac{65520}{691} \sum_{m \geq 1} \sigma_{11}(m) q^m, \\ F &= (2\pi)^{-12} \Delta = q \prod_{m \geq 1} (1 - q^m)^{24} = \sum_{m \geq 1} \tau(m) q^m. \end{aligned}$$

Thus we can write our theta function as

$$\theta_{\Lambda_{24}} = E_{12} + c_{\Lambda_{24}} F,$$

for a constant $c_{\Lambda_{24}}$.

Comparing coefficients gives us

$$r_{\Lambda_{24}}(m) = \frac{65520}{691} \sigma_{11}(m) + c_{\Lambda_{24}} \tau(m).$$

We find $c_{\Lambda_{24}}$ by plugging in $m = 1$:

$$c_{\Lambda_{24}} = \underbrace{r_{\Lambda_{24}}(1)}_{=0} - \frac{65520}{691} \underbrace{\tau(1)}_{=1} = -\frac{65520}{691}$$

Example. We can now, for example, calculate the number of points of length 2 in the Leech lattice.

Plug in $m = 2$ (because $\sqrt{2 \cdot 2} = 2$) in $r_{\Lambda_{24}}(m)$ and get

$$\frac{65520}{691}(\underbrace{\sigma_{11}(2)}_{=2049} - \underbrace{\tau(2)}_{=-24}) = 196560.$$

Formulated the other way around, from all the above discussion, we find the congruence

$$\sigma_{11}(m) \equiv \tau(m) \mod 691$$

5.2 Λ_8-Lattice

Here again the definition of this lattice

$$\Lambda_8 = \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8 : \sum_i x_i = 0 \mod 2\}$$

As discussed, it is unimodular, moreover it is an even lattice as well and the only such lattice of dimension 8.

Let's calculate its theta function (this time a modular form of weight 4) as well:

$$\theta_{\Lambda_8} = \sum_{m \geq 0} r_{\Lambda_8}(m) q^m$$

From last section's corollary, we find now

$$\theta_{\Lambda_8} = E_4 + f_{\Lambda_8}$$

but every cusp form of weight 4 is the zero-function, so we get

$$r_{\Lambda_8}(m) = 240\sigma_3(m).$$